ELEMENTARY GEOMETRY.

Cambridge:
PRINTED BY C. J. CLAY, M.A.
AT THE UNIVERSITY PRESS.

ELEMENTARY GEOMETRY

BOOKS I. II. III.

ANGLES, PARALLELS, TRIANGLES, EQUIVALENT FIGURES, THE CIRCLE, AND PROPORTION.

BY

J. M. WILSON, M.A.

LATE FELLOW OF ST JOHN'S COLLEGE, CAMBRIDGE,
AND MATHEMATICAL MASTER OF RUGBY SCHOOL.

SECOND EDITION.

London and Cambridge:
MACMILLAN AND CO.
1869

PREFACE TO THE SECOND EDITION,

CONTAINING BOOKS I, II, III.

THE experience that has been gained by those who have used this book in Rugby and in other Schools has enabled me to make some important improvements in this edition. The first two sections have been entirely rewritten, and will be found easier and more exact. The Exercises in the earlier sections have been increased in number.

Book II. will, I hope, justify the remarks in pp. xii, xiii of the Preface to the first edition.

Since this Geometry was written several similar works have appeared in England, and the use of them appears to be spreading, as schoolmasters recognize their merits as text-books. Cambridge, I regret to say, continues to demand Euclid and nothing but Euclid.

RUGBY,
February, 1869.

PREFACE TO THE FIRST EDITION OF BOOK I.

The lately published Report of the Schools Inquiry Commission has given an immediate importance to the question whether Euclid's Elements is the proper text-book for teaching Geometry to beginners. Euclid's recognised and acknowledged faults as a system of Geometry, and as a specimen of analysed reasoning, are of slight importance compared with others of greater magnitude. The real objections to Euclid as a text-book are his artificiality, the invariably syllogistic form of his reasoning, the length of his demonstrations, and his unsuggestiveness.

As to the first, he aimed, not at unfolding Geometry as a science, but at shewing on how few axioms and postulates the whole could be made to depend: and he has thus sacrificed, to a great extent, simplicity and naturalness in his demonstrations, without any corresponding gain in grasp or cogency. The exclusion of hypothetical constructions may be mentioned as a self-imposed restriction which has made the confused order of his first book necessary, with

out any compensating advantage. There is no real advantage in the arrangement of propositions by sequence which Euclid maintains so strictly, for nothing can be gained by excluding *any* sound method of reasoning: and if a direct proof can be found of any theorem, which naturally arises out of the data of the theorem, it is to be preferred to a circuitous proof which depends on other theorems. Thus if the Fifth Proposition can be proved independently, and on its own evidence, it is certain that the decisive bearing of the data on the conclusion will be better appreciated than it would be on Euclid's method.

Again, as his self-imposed restrictions, geometrical and logical, have made his Geometry confused in its arrangement, and unnatural and forced in the nature of his proofs, so too the detailed syllogistic form into which he has thrown all his reasonings, is a source of obscurity to beginners, and damaging to true geometrical freedom and power.

We put a boy down to his Euclid; and he reasons for the first time, a task in itself difficult enough; but we make him reason in iron fetters. There is, of course, a natural and inevitable difficulty in the task of tracing data into the inferences founded on them; the geometrical facts are new: it is new to the learner to find himself reasoning consecutively at all. If then to all this novelty we add the constant analysis into syllogisms of inferences which are obvious without this analysis, and the constant reference to general axioms and general propositions, which are no clearer in the general statement than they are in the particular instance, we make

the study of Geometry unnecessarily stiff, obscure, tedious and barren. Many facts perfectly familiar to common sense and experience become strange and unrecognizable when thus expressed; and conclusions the most obvious become cloudy by the extreme detail of the proofs. And the result is, as every one knows, that boys may have worked at Euclid for years, and may yet know next to nothing of Geometry.

The length of his demonstrations is a real evil; for, when a geometrical theorem has become familiar, it is easy to analyse into a demonstration the perception that we have obtained of the certainty of the theorem, but until that familiarity is gained great length in demonstration exercises the memory more than the intelligence. Messrs. Demogeot and Montucci in their Report on English Education begin one of their chapters with the following remarkable words: "Le trait distinctif de l'enseignement des mathématiques en Angleterre c'est qu'on y fait appel plutôt à la mémoire qu'à l'intelligence de l'élève." And they remark later on, "On y trouve (in Euclid) sans doute une logique de fer, qui n'admet point de réplique; mais aussi arrive-t-on aux résultats les plus évidents par un verbiage absolument en désaccord avec nos élégantes habitudes d'une concision non moins vigoureuse que la prolixité d'Euclide." And unquestionably one result of the tediousness of Euclid is that so little knowledge of geometry is gained; so little, there is abundant evidence to prove, that our education is more marked by inferiority to other nations in this respect than in any other.

And again, unsuggestiveness is a great fault in a text-book. Euclid places all his theorems and problems on a level, without giving prominence to the master-theorems, or clearly indicating the master-methods. He has not, nor could he be expected to have, the modern felicity of nomenclature. The very names of *superposition*, *locus*, *intersection of loci*, *projection*, *comparison of triangles*, do not occur in his treatise. Hence there already exists a wide gulf between the form in which Euclid is read, and that in which he is generally taught. Unquestionably the best teachers depart largely from his words, and even from his methods. That is, they use the work of Euclid, but they would teach better without it. And this is especially true of the application to problems. Everybody recollects, even if he have not the daily experience, how unavailable for problems a boy's knowledge of Euclid generally is. Yet this is the true test of geometrical knowledge; and problems and original work ought to occupy a much larger share of a boy's time than they do at present.

On the other side there will be brought two arguments, and in general two arguments alone.

It will be urged that if Euclid is given up we shall lose the advantage of uniformity. Its place will be taken by scores of manuals of Geometry, and examinations in Geometry will become impossible. On such a point it will be difficult to persuade others that the advantage is nearly imaginary. The fact is, that Geometry when treated as a science, treated inartificially, falls into a certain order from

which there can be no very wide departure; and the manuals of Geometry will not differ from one another nearly so widely as the manuals of algebra or chemistry: yet it is not difficult to examine in algebra and chemistry. The experience of the French schools and examinations is decisive on this point. I am assured that they experience no such difficulty in examining as we might imagine. It is true that by the loss of Euclid less weight will be given to bookwork, and more to problems, in elementary Geometry; but surely this will not be thought an evil. And to allow some latitude of expression is to admit different degrees of elegance, accuracy and perspicuity, and cannot fail to encourage these merits in the style of the student.

Secondly, it is commonly said that Euclid is taught not so much for the sake of his Geometry as for the sake of his Logic.

It is most important that there should be no confusion of ideas on this point.

Geometry is taught on several grounds; one of which is that it forms an excellent subject-matter for continuous reasoning and clear demonstration; and the merit of Euclid's treatment of Geometry consists in his resolutely fixing the mind on the *things* with which the argument is concerned, and never allowing the substitution of mere *symbols.* Algebra fails as a discipline because it speedily degenerates into a manipulation of symbols: in Euclid every step is planted on firm ground. Geometry treated algebraically would lose this merit altogether; but it is not

proposed to treat Geometry algebraically. I contend that it is possible to present the science of Geometry in a more natural and simple order, and demonstrate the properties of figures by proofs differing in many cases from Euclid's, without being less rigorous.

And when it is urged that Euclid affords an excellent training in logic, it must not be forgotten that very heavy deductions have to be made from this argument.

For the test of logical training is the power of applying the same processes of reasoning to new matter. If a boy has learnt logic, he can use his logic. It is a mere delusion to believe that boys or men have got a logical training from the study of Euclid if they are unable to solve a problem or work a deduction. Their recollection of Euclid may be perfect, but the test of training is the *power* it has given; and if boys can produce no original work, the training is assuredly worth very little. The fact is, Euclid is a beautiful exercise in logic to a mathematician; but the mathematics must in many cases be learnt first, or the training is hopelessly out of the learner's reach.

Again, the artificial restrictions in Euclid are a serious drawback to his value as a logical training as well as to his system of Geometry. Many students of Euclid are incapable of distinguishing between a failure in a mathematical proof and in the observance of a conventional rule. Can it be called a good training in logic which obscures the distinction between what is artificial and what is necessary? Do not many mathematicians, trained in Euclid,

say that trisecting an angle or squaring a circle is a *Geometrical* impossibility, from this confusion of ideas? And again, Euclid's treatment of parallels distinctly breaks down in Logic. It rests on an axiom which is not axiomatic. It is often said that mathematicians shew more anxiety about their reasoning than about their premises. Whether this be so or not, mathematicians have Euclid's example to plead.

There is moreover a logic besides that of mere reasoning. It is a most valuable discipline in logic to train the mind to scientific order; to teach the relation of general truths to each other, and to the particular cases which fall under them; to make science light its own way by means of a natural arrangement. Now all this cannot be taught by Euclid at all, and might be taught by a well-arranged Geometry.

In short, the logical training to be got from Euclid is very imperfect, and in some respects bad; and yet to this imperfect logical training, such as it is, we confessedly sacrifice our Geometrical teaching: while it is possible to arrange Geometry in a lucid order, and adhering to rigorous methods of demonstration, teach the logic as inseparable from the science.

The very parts of Euclid's Geometry which are most admired by mathematicians, for their ingenuity and completeness, as for example his definition of Proportion in Book v., are open to fatal objections. "Common sense," says De Morgan, "requires that we should satisfy the notion of proportion existing in the mind previously to

entering on Geometry as evinced by sameness of relative magnitude, and not invent a new notion for the occasions of Geometry." The reasoning is exquisite and profound; it is too exquisite; it leaves on most men's minds the half-defined impression that all profound reasoning is something far-fetched and artificial, and differing altogether from good clear common sense.

In common then with some of our ablest mathematicians, and with many who are engaged in teaching mathematics, I am of opinion that the time is come for making an effort to supplant Euclid in our schools and universities. Already the fifth book has practically gone; and in consequence the study of the sixth book has become somewhat irrational.

For the improvement of our Geometrical teaching in England two things seem to be wanted. First, that Cambridge, and the Government Examiners, should follow the example of Oxford and London, and examine not in Euclid only, but in the Geometry of specified subjects, according to a programme for each examination. Secondly, that text-books should be written to illustrate what is required. This book is the first part of such a text-book of Elementary Geometry. Probably its method of treating propositions may be considered as sufficiently different from that common in England to justify its publication. At the same time it may be remarked, that the forthcoming second part, which is to embrace the Geometry of the Circle and the applications of proportion to Plane Geometry, ought to

bring out in a stronger light than is here possible the superiority of modern to ancient methods of Geometry. The present Part may be used either as introductory to Euclid, or as replacing the first two books. In a few years I hope that our leading mathematicians will have published, perhaps in concert, one or more text-books of Geometry, not inferior, to say the least, to those of France, and that they will supersede Euclid by the sheer force of superior merit.

In the compilation of this little book, I lay claim to no originality. I have read several French Geometries, and am under some obligations to them. I owe more to the valuable suggestions of my colleague the Rev. C. E. Moberly, who has the spirit without the prejudices of a geometrician. But much of what is most characteristic in the book is due to Dr Temple. It was at his wish that I undertook the work, as he is strongly impressed with the need of it; and his criticisms and his contributions to it have enabled me to rearrange it and improve it in some important respects. And this gives me confidence in publishing it. At the same time we feel that the experience of a few years in teaching with this book will enable me to make improvements in it, without departing widely from the lines here laid down. To Professor Hirst also I have the pleasure to express my thanks for some corrections and remarks.

The distinctive features of this work are intended to be the following. The classification of Theorems according

to their subjects; the separation of Theorems and Problems; the use of hypothetical constructions; the adoption of independent proofs where they are possible and simple; the introduction of the terms *locus*, *projection*, *&c;* the importance given to the notion of direction as the property of a straight line; the intermixing of exercises, classified according to the methods adopted for their solution; the diminution of the number of theorems; the compression of proofs, especially in the latter part of the book; the tacit, instead of explicit, reference to axioms; and the treatment of parallels.

J. M. WILSON.

RUGBY,

April 28, 1868.

CONTENTS.

BOOK I.

SECTION I. THE STRAIGHT LINE AND ANGLE.

SECTION II. PARALLELS.

SECTION III. TRIANGLES.

SECTION IV. PARALLELOGRAMS.

SECTION V. PROBLEMS OF CONSTRUCTION.

SECTION VI. THE EQUIVALENCE OF FIGURES.

BOOK II. THE CIRCLE.

SECTION II. ANGLES IN SEGMENTS OF THE CIRCLE.

SECTION III. THE TANGENT AND NORMAL.

SECTION IV. PROBLEMS.

SECTION V. REGULAR POLYGONS.

BOOK III. PROPORTION.

SECTION I. APPLICATION OF PROPORTION TO LINES.

SECTION II. AREAS.

SECTION III. PROBLEMS.

ELEMENTARY GEOMETRY.

INTRODUCTION.

THE notions of a *solid*, a *surface*, a *line*, are obtained by experience from external objects. The science of *Geometry* deals with the properties and construction of *figures*, that is of solids, surfaces, and lines considered independently of the substances of which they are composed. *Plane* Geometry deals only with the line and *plane*, or flat surface; and the *elements* of plane Geometry include the properties only of the straight line and circle, and of combinations of straight lines and circles.

The Science of Geometry is *deductive*, certain common notions being assumed, and shewn to lead necessarily to those results which constitute the science. These common notions are called *axioms*, which are therefore *truths admitted without demonstrations*. Some of these are geometrical; others are applicable to all kinds of magnitude. The first should be stated explicitly, when they are first assumed; but the second are not peculiar to Geometry, and need not be stated explicitly.

A Theorem is a Geometrical truth which is capable of demonstration by reasoning from certain known truths.

A Corollary to a theorem is a geometrical truth easily deducible from that theorem.

The sign = is used to represent equality. Thus $A = B$ is written to express that A is equal to B.

The signs $>$ $<$ are used to express 'is greater than,' 'is less than:' thus $C > D$ expresses that C is greater than D.

The sign + is used to express addition, and − to express subtraction.

The sign $\therefore$ is used for *therefore*, and $\because$ for *because*.

BOOK I.

In the present Book we shall treat only of certain properties of straight lines.

SECTION I. THE STRAIGHT LINE AND ANGLE.

Def. 1. A *Surface* is the boundary of a solid.

Def. 2. A *geometrical line* has position, and length, and at every point of it it has direction, but differs from a physical line in being considered as not having breadth or thickness.

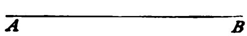
The boundaries of a surface are *lines*.

Def. 3. A *geometrical point* has position, and differs from a physical point in being considered as not having magnitude.

The intersection of two geometrical lines, and the extremities of a geometrical line are *points*.

When lines and points are spoken of it will be understood that *geometrical* lines and points are intended.

Def. 4. A *straight line* is a line which has the *same* direction at all parts of its length.



It has also the opposite direction; thus the direction ot *AB* is opposite to that of *BA*.

A straight line may be conceived as generated by a point moving always in the *same* direction.

Other lines are called *broken*, *curved*, or *mixed*, as

These lines may be conceived as generated by a point which moves in a direction which is altered suddenly or gradually.

Def. 5. A *plane surface* is one in which any two points being taken, the straight line which joins them lies wholly in that surface.

Hence the practical test of a plane surface is that the straight edge of a ruler shall coincide with it in all positions.

Axioms respecting straight lines.

Ax. 1. That a straight line marks the shortest distance between any two of its points.

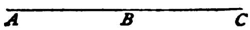
Ax. 2. That if two straight lines have two points in common they coincide wholly.

Ax. 3. A line may be conceived as transferred from any position to any other position, its magnitude being unaltered.

A straight line may be found equal to the ***sum*** of two lines by placing them in the same straight line with a common extremity, so that a point which by its motion has described the one, would, if its motion were ***continued in the same direction***, immediately proceed to describe the other.

The operation of adding is expressed by the sign +.

Thus $AB + BC = AC$.



Similarly, the ***difference*** of two lines may be found by placing them in the same straight line, with a common extremity, so that a point which has described the one, would, if its motion were ***reversed***, immediately proceed to describe the other.

The operation of subtracting is expressed by the sign −.

Thus $$AC - BC = AB.$$

The sign ~ placed between two quantities means that the less is to be subtracted from the greater; thus

$$AB \sim AC = BC.$$

Lines are expressed arithmetically as multiples of some known length. Thus we speak of 7 yards, 11 miles, a yard and a mile being known lengths.

Exercises on Straight Lines.

1. Prove that the sum and difference of two lines are together double of the greater of them.

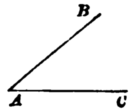
2. AB is divided into two equal parts in O, and P is any point in the line AB: prove that $\frac{1}{2}(AP \sim BP) = OP$.

3. On the same supposition if P is any point in AB produced through A or B, prove that $\frac{1}{2}(AP + BP) = OP$.

Angles.

Def. 6. Two straight lines that meet one another form an *angle* at the point where they meet: and the lines are called the *arms* of the angle; and the point the *vertex* of the angle.

An angle is a magnitude, and is measured by the *quantity of turning* that one of its arms must undergo in order to be brought to coincide with the other.



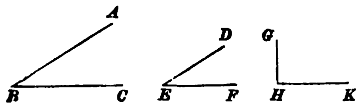
It is obvious that the magnitude of an angle does not depend on the length of its arms.

An angle may be conceived as generated by the rotation of an arm round its extremity, the motion being in one plane.

An angle is named by a single letter at the vertex, as A; or by a letter at the vertex placed between letters on each of the arms.

Ax. 4. An angle may be conceived as transferred to any other position, its magnitude being unaltered.

Ax. 5. Angles are *equal* when they could be placed on one another so that their vertices would coincide in position, and their arms in direction.



Thus the angles B, E are equal, if when E is placed on B, and EF on BC, then ED has the same direction as BA.

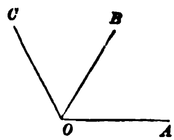
Conversely, angles which are equal can be conceived as placed on one another so as to coincide.

And angles which are unequal could not coincide.

The *sum* of two angles is found by placing them so as to have a common vertex and a common arm, and on opposite sides of the common arm, so that an arm which by its rotation has described one angle, would, if its motion were continued in the *same* direction, immediately proceed to describe the other.

Angles so placed are said to be *adjacent*.

The *difference* of two angles is found by placing them on the same side of the common arm, so that the arm which has described the first angle would, if its motion were *reversed*, immediately proceed to describe the other.



Thus AOC is the sum of the angles AOB, BOC; and AOB is the difference of AOC and BOC.

This is expressed by saying

$$AOC = AOB + BOC$$

and $$AOB = AOC - BOC.$$

Def. 7. The *bisector* of an angle is the line that divides it into two equal angles. It is obvious that an angle can have one, and only one bisector. Thus if the angle $AOB =$ the angle BOC, then OB is the bisector of the angle AOC.

Exercises on Addition and Subtraction of Angles.

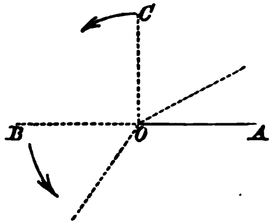
1. Prove that if an angle AOC is bisected by OB, and divided into two unequal angles by OP, then

$$AOP \sim COP = 2\ BOP.$$

2. Prove that if an angle AOC is bisected by OB, and a line OP is drawn outside the angle AOC, then

$$AOP + COP = 2\ BOP.$$

3. The sum and difference of two angles are together equal to twice the greater angle.



Def. 8. A line may be conceived to revolve continuously in one direction round its extremity until it returns once more to its initial position.

It is then said to have made *one revolution*.

Def. 9. When it coincides with what was initially its continuation, it has described *half a revolution*, and the angle it has then described is called *a straight angle*, because the arms of it form one straight line.

Thus the angle AOB is a straight angle. All straight angles are equal to one another. *Ax.* 2 and 5.

Def. 10. Half a straight angle, or a quarter of one revolution, is called *a right angle*.

Thus if AOC and COB are equal, each of them is a right angle, or half a straight angle, or a quarter of a revolution.

All right angles are therefore equal to one another.

Def. 11. A straight line is said to be *perpendicular* to another straight line when it makes a right angle with it.

Hence there can be only one perpendicular to a given line at a given point, on one side of that line, because only one line can make a right angle with the given line at that point.

Def. 12. An *acute angle* is one which is less than a right angle.

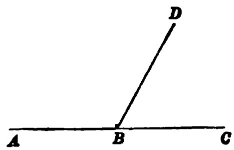
Def. 13. An *obtuse angle* is one which is greater than a right angle.

Def. 14. A *reflex angle* is one which is greater than a straight angle.

THEOREM I.

When a straight line stands upon another straight line, it will make the adjacent angles together equal to two right angles: and conversely, if two adjacent angles are equal to two right angles, the exterior arms of these angles will be in the same straight line.

Let DB stand upon the straight line AC; then will the adjacent angles ABD, DBC be together equal to two right angles.



Proof. For the sum of ABD and DBC is the straight angle ABC, which is equal to two right angles. Def. 10.

Conversely, if ABD and DBC are together equal to two right angles, AB and BC will be in one straight line.

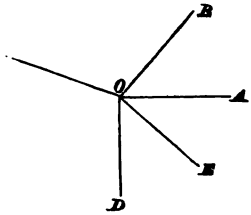
Proof. For ABD and DBC make up a straight angle by supposition, and therefore the arms of it, BA and BC, will be in one straight line. Def. 9.

Def. 15. Two angles are said to be *supplementary* to one another when their sum is a straight angle.

Def. 16. Two angles are said to be *complementary* to one another when their sum is a right angle.

THEOREM 2.

The sum of all the angles made by any number of lines taken consecutively which meet at a point will be four right angles.



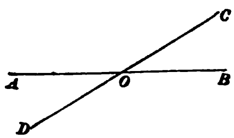
Let any number of lines OA, OB, OC, OD, OE meet at O, then $AOB+BOC+COD+DOE+EOA$ will make up four right angles.

Proof. For the sum of these angles is an angle of one revolution, which is equal to four right angles.

Def. 17. The opposite angles made by the intersection of two straight lines are called *vertically opposite angles.*

THEOREM 3.

Vertically opposite angles will be equal to one another.



Let AOD, BOC be vertically opposite angles. They will be equal.

Proof. For since the same quantity of turning of AB round O which would make OB coincide with OC, would also make OA coincide with OD, it follows that the angle AOD = the angle BOC.

Similarly $AOC=BOD$.

Def. 18. Angles are expressed *arithmetically* as multiples of some known angle. For this purpose a right angle is divided into 90 equal angles which are called *degrees*, and written thus, 45°. A degree is subdivided into 60 equal parts called *minutes* and written thus, $15'$. And a minute is subdivided into 60 *seconds* ($60''$).

Exercises on Angles.

1. If two straight lines intersect at a point, and one of the four angles is a right angle, prove that the other three are right angles.

2. If five lines meet at a point and make equal angles with one another all round that point, each of the angles will be four-fifths of a right angle. Express this in degrees.

3. If the four angles made by four straight lines which meet at a point are all right angles, prove that the four lines form two straight lines.

4. *Two* straight lines meet at a point. Are the angles at that point together equal to four right angles?

5. Prove that the bisectors of adjacent supplementary angles are at right angles to one another.

6. Find the angle between the bisectors of adjacent complementary angles.

7. Of two supplementary angles the greater is double of the less; find what fraction the less is of four right angles.

8. Twelve lines meet at a point so as to form a regular twelve-rayed star: find the number of degrees in the angle between consecutive rays.

9. If A is the number of degrees in any angle, prove that $90^\circ + A^\circ$ is supplementary to $90^\circ - A^\circ$; and that $45^\circ + A^\circ$ is complementary to $45^\circ - A^\circ$.

10. Find the supplement and complement of $21^\circ\ 35'\ 45''$.

11. If four straight lines OA, OB, OC, OD meet at a point, and $AOB = COD$, and $BOC = DOA$, prove that AOC, BOD are straight lines.

12. Prove that the bisectors of the four angles which one straight line makes with another form two straight lines at right angles to one another.

SECTION II. PARALLELS.

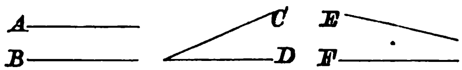
In the first section a single straight line was defined as a line which has the same *direction* at all parts of its length.

We now proceed to consider the relations of two or more straight lines in one plane in respect of direction.

Ax. 6. Two different straight lines may have either the same or different directions.

Ax. 7. Two different straight lines which meet one another have different directions.

Ax. 8. Two straight lines which have different directions would meet if prolonged indefinitely.



Thus A and B in the figure have the same direction; and C and D which meet have different directions; and E and F which have different directions would meet if produced far enough.

Def. 19. Straight lines which are not parts of the same straight line, but have the same direction, are called *parallels.*

From this definition, and the axioms above given, the following results are immediately deduced:

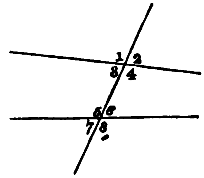
(1) That parallel lines would not meet however far they were produced.

For if they met, they would have different directions by *Ax.* 7.

(2) That lines which are parallel to the same line are parallel to one another.

For they each have the same direction as that line, and therefore the same direction as the other.

Def. 20. When a straight line intersects two other straight lines it makes with them eight angles which have received special names.



In the figure 1, 2, 7, 8, are called *exterior* angles, and 3, 4, 5, 6, are called *interior* angles.

Again, 1 and 5 are said to be *corresponding* angles; so also are 2 and 6, 7 and 3, 8 and 4: and 3 and 6 are said to be *alternate* angles, so also are 4 and 5.

Ax. 9. An angle may be conceived as transferred from one position to another, the direction of its arms remaining the same.

Theorem 4.

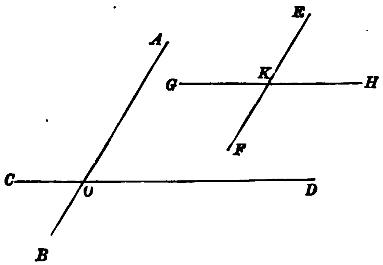
If two lines are respectively parallel to two other lines, the angles made by the first pair will be equal or supplementary to the angles made by the second pair; equal, if both are taken in the same or both in the opposite direction; supplementary, if one is taken in the same and one in the opposite direction.

Let AOB, COD be respectively parallel to EKF, GKH.

Then will the angle AOD be equal to EKH or GKF, and supplementary to EKG and HKF.

Proof. For conceive the angle AOD transferred to K,

the direction of its arms being unaltered. *Ax.* 9. Then since, by hypothesis, OA and OD have the same direction as KE and KH, they would then coincide with KE and KH; and the angle AOD would coincide with the angle EKH.



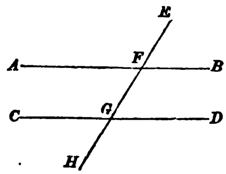
Therefore the angle $AOD = EKH$.

But $EKH = GKF$ (Th. 3), and therefore $AOD = GKF$.

Also GKE or FKH is supplementary to EKH (Th. 1), and therefore GKE or FKH is supplementary to AOD.

COR. *If a straight line intersect two parallel lines, it will make the corresponding angles equal, the alternate angles equal, and the interior angles on the same side of the intersecting line supplementary.*

Let AB, CD be parallels, and let $EFGH$ intersect them.



Then will the angles at G be equal to the corresponding angles at F.

Proof. For conceive the angles at G, transferred to F, the direction of the lines being unaltered. Each angle would then coincide with its corresponding angle, and is therefore equal to it.

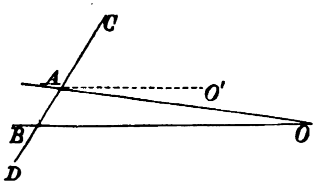
Also AFG is equal to EFB (Th. 3), and therefore equal to FGD, which is its alternate angle.

And BFG is supplementary to EFB (Th. 1), and therefore also to FGD.

Theorem 5.

If two straight lines meet in a point they will make unequal corresponding angles with every straight line which meets them.

Let AO and BO meet at O, and be intersected by any straight line CD, in the points A and B. Then will the corresponding angles CAO, CBO be unequal.



Proof. Then since AO and BO meet they have different directions (Ax. 7), and therefore if the angles B were transferred to A, the direction of their arms being unaltered, the arm BO would not coincide with AO, while BA would coincide with AC.

Hence the angle CAO is not equal to the corresponding angle CBO.

Cor. 1. *Hence if two straight lines which are not parallel are intersected by a third, the alternate angles will be not equal, and the interior angles on the same side of the intersecting line will be not supplementary.*

Cor. 2. *Hence also if the corresponding angles are equal, or the alternate angles equal, or the interior angles supplementary, the lines will be parallel.*

For they cannot be not parallel, for then the corresponding and alternate angles would be unequal by Cor. 1.

Cor. 3. *From a given point outside a given line only one perpendicular can be let fall on that line.*

Remark. Theorems may often be arranged in groups of four; an original theorem, its opposite, its converse, and the converse of its opposite. Thus, "If A is B, then C will be D," may be taken as the type of a theorem. "If A is B" is the *hypothesis* or supposition; "C will be D" is the *conclusion.*

The *opposite* of a theorem is formed by negativing the hypothesis and conclusion. Thus, "If A is not B, then C will not be D," is the opposite of the original theorem.

The *converse* of a theorem is formed by transposing the hypothesis and conclusion. Thus, "If C is D, then A will be B" is the converse. And obviously, "If C is not D, then A will not be B" is the *converse of the opposite*, or the *opposite of the converse.*

It must be observed that if two of these theorems are true, then the rest follow logically.

Thus, "If two lines are parallel, the corresponding angles will be equal" is the original theorem; its opposite is, "If two lines are not parallel, the corresponding angles will be unequal." Hence the converse theorem follows that "if the corresponding angles are equal the lines will be parallel," and the converse of the opposite, that "if the corresponding angles are unequal the lines will not be parallel."

Def. 21. A figure enclosed by any number of straight lines is called a *polygon.*

It is called *convex* when no one of its angles is reflex.

It is called *regular* when it is equilateral and equiangular, that is when all its sides and angles are equal.

A regular four-sided figure is called a *square.*

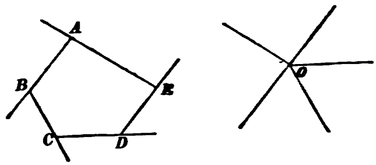
The line joining any two angles not adjacent is called a *diagonal.*

When the number of its sides is 3, 4, 5, 6..., it is called a *triangle*, a *quadrilateral*, a *pentagon*, a *hexagon*, and so on.

Theorem 6.

The exterior angles of any convex polygon, made by producing the sides in succession, will be together equal to four right angles.

Let $ABCDE$ be a polygon having all its sides produced in succession; then will the sum of its exterior angles be equal to four right angles.



Proof. For conceive lines drawn through any point O parallel to the sides of the polygon, in the directions in which the sides are produced.

Then (by Theorem 4) the angles taken consecutively round O are equal to the exterior angles of the polygon. But the angles at O are together equal to four right angles (Th. 2). Therefore the exterior angles of the polygon are together equal to four right angles.

Cor. *Hence it may be shewn that all the interior angles of any polygon are less than twice as many right angles as the figure has sides by four right angles.*

Proof. For each interior angle with its adjacent exterior angle = two right angles (Th. 1);

Therefore all the interior angles + all the exterior angles = twice as many right angles as the figure has sides;

But all the exterior angles = four right angles (Th. 6);

Therefore all the interior angles + four right angles = twice as many right angles as the figure has sides;

and therefore all the interior angles are less than twice as many right angles as the figure has sides by four right angles.

This Corollary leads to many interesting results.

Let it be required, for example, to find the magnitude of the angle of a regular hexagon. Since the six interior angles are less than twelve right angles by four right angles, they are together equal to eight right angles; and therefore each of them is eight-sixths of a right angle, or is $1\frac{1}{3}$ of a right angle, or contains 120°.

Exercises.

1. Prove that the angles of any triangle are together equal to two right angles.

2. Shew that the angles of an equiangular triangle are equal to two-thirds of a right angle. Express them in degrees.

3. Find the magnitude of the angle of a regular octagon.

4. How many equiangular triangles can be placed so as to have one common angular point, and fill up the space round it?

5. Shew that three regular hexagons can be placed so as to have a common point, and fill up the space round that point.

6. Shew that two regular octagons and one square have the same property.

Draw a pattern consisting of octagons and squares.

7. Shew that the angle of a regular pentagon is to the angle of a regular decagon as 3 to 4.

8. If a line is perpendicular to another it will be perpendicular to every line parallel to it.

9. If a polygon is equilateral, does it follow that it is equiangular, and conversely?

10. Shew that the exterior angle of a regular polygon of n sides is $\frac{1}{n}$ of $360°$: and that its interior angle is $\left(2-\frac{4}{n}\right)90°$.

11. How many diagonals can be drawn in a pentagon? How many in a decagon?

12. Shew that a square, a hexagon and a dodecagon will fill up the space round a point; and make a pattern of these polygons.

13. Examine whether a square, a pentagon and an icosagon have the same property; and also whether a pattern can be constructed of pentagons and decagons.

14. The exterior angle of a regular polygon is one-third of a right angle: find the number of sides in the polygon.

15. Two lines intersecting in A are respectively perpendicular to two lines intersecting in B: prove that any angle at A is equal or supplementary to any angle at B.

16. The arms of an angle A are equally inclined, in the same direction of rotation, to the arms of an angle B. Prove that the angle A is equal to the angle B.

SECTION III. TRIANGLES.

Def. 22. *A triangle* is the figure contained by three straight lines, which are called its *sides*. The sum of the sides is sometimes called the *perimeter*.

Def. 23. A triangle is called *isosceles* when two of its sides are equal.

Def. 24. A triangle is called *equilateral* when all its sides are equal, and *equiangular* when all its angles are equal.

Def. 25. It is said to be *right-angled* or *obtuse-angled* when one of the angles is a right angle or an obtuse angle, and *acute-angled* when all its angles are acute angles.

Def. 26. A triangle is sometimes regarded as standing on one of its sides, which is then called its *base;* and the angle opposite that side is called the *vertex*. When two of the sides of a triangle have been mentioned, the remaining side is often spoken of as the *base*.

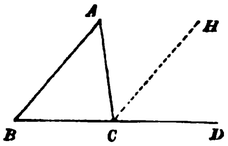
Def. 27. The term *hypothenuse* is used to describe the side of a right-angled triangle which is opposite to the right angle.

Def. 28. The *area* of a figure is the space enclosed by the sides of the figure.

Def. 29. Triangles are said to be *equal in all respects* when they have the three sides and the three angles of the one equal to the three sides and the three angles respectively of the other, and the area of one equal to the area of the other.

Ax. 10. A triangle may be conceived as taken up and placed in any other position, the magnitude of its parts remaining unaltered.

THEOREM 7.



If one side of a triangle is produced, the exterior angle will be equal to the two interior and opposite angles, and the three interior angles of any triangle will be together equal to two right angles.

Let the side BC of the triangle ABC be produced to D: then shall the angle $ACD =$ the sum of the angles ABC, CAB. And the three angles ABC, BCA, CAB shall be together equal to two right angles.

Proof. For if through C a line CH were drawn parallel to BA,

the angle $HCD =$ the corresponding angle ABC (Th. 4),
and the angle $ACH =$ the alternate angle BAC (Th. 4);
$\therefore$ the whole angle $ACD =$ the two angles $ABC + BAC$.

And if ACB be added to these,
the two angles $ACD + ACB =$ the three angles $ABC + BCA + CAB$.

But $ACD + ACB =$ two right angles (Th. 1);
therefore $ABC + BCA + CAB =$ two right angles.

Remark. This latter result is obviously a particular case of the preceding Theorem.

COR. 1. *It follows that no triangle can have more than one right angle or obtuse angle.*

COR. 2. *In a right-angled or obtuse-angled triangle the right or obtuse angle is the greatest angle.*

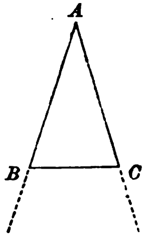
COR. 3. *In any right-angled triangle the two acute angles together make up one right angle.*

COR. 4. *An exterior angle of a triangle is greater than either of the interior and opposite angles.*

The four next Theorems are properties of a single triangle.

THEOREM 8.

An isosceles triangle will have the angles at its base equal.



Let ABC be an isosceles triangle, having $AB = AC$.

Then shall the angle $B =$ the angle C.

Proof. For conceive the triangle ABC taken up and put down in its former position, with the arms of the angle A transposed. (Ax. 10.) That is, let A be put down where it was before, and AC where AB was; then AB would be where AC was, since the angle at A is unchanged, and B would fall where C was, and C where B was, because $AB = AC$; and BC would fall as it was before, but in a reversed position; that is, the angle B would fall on the angle C, and therefore the angle $B =$ the angle C.

COR. 1. *The angles on the other side of the base, made by producing the equal sides, will be equal.*

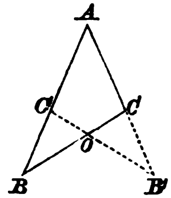
For they are respectively supplementary to the angles at the base (Th. 1).

COR. 2. *An equilateral triangle will be equiangular.*

For every pair of the angles is equal.

THEOREM 9. Oppositely.

A triangle which is not isosceles will have the angles at its base unequal, the greater angle being opposite to the greater side.



Let ABC be a triangle in which AB is greater than AC.

Then will the angle ACB be greater than the angle ABC.

Proof. Conceive the triangle BAC taken up and put down with the arms of

the angle at A transposed: that is, let AC fall as AC', and AB as AB', and therefore BC as $B'C'$; and let BC, $B'C'$ intersect in O.

Then the angle ACB is the same as the angle $AC'B'$; but $AC'B'$ is greater than ABC, because it is the exterior angle of the triangle $OC'B$ (Th. 7, Cor. 4); and therefore ACB is greater than ABC.

THEOREM 10. Conversely.

A triangle which has the angles at its base equal will be isosceles.

Let the triangle ABC have the angles B and C equal.

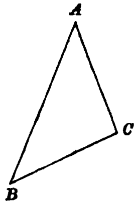
Then shall $AB = AC$.

Proof. For AB is not unequal to AC, for then the angle C would be unequal to the angle B (Th. 9), which it is not and therefore $AB = AC$.

COR. *An equiangular triangle is equilateral.*

THEOREM 11. Conversely to the opposite.

A triangle which has two of its angles unequal shall have the sides opposite them unequal, the greater side being opposite to the greater angle.



Let the angle ACB be greater than the angle ABC, then shall AB be greater than AC.

Proof. For AB is not equal to AC, for then (by Th. 8) the angle ABC would be equal to the angle ACB.

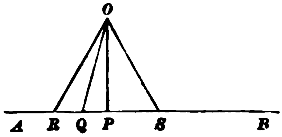
Nor is AB less than AC, for then the angle ACB would be less than ABC (by Th. 9);

Therefore AB is greater than AC.

THEOREM 12.

Of all the straight lines that can be drawn from a given point to meet a given straight line, the shortest will be the perpendicular; and of the others, that which is further from the perpendicular will be greater than that which is nearer; and one, and only one, oblique can be drawn equal to any given oblique, and it will be on the other side of the perpendicular and equally inclined to it.

Let O be the given point, AB the given straight line. Then (by Theorem 5, Cor. 3) there can be only one line OP perpendicular to AB.



Let OQ be any other line from O meeting AB in Q; OP shall be less than OQ.

Proof. For since OPQ is a right angle, OQP is an acute angle (Th. 7, Cor. 2); that is, $OPQ > OQP$,

and therefore $\qquad OQ > OP$ by Theorem 11.

Hence OP is less than *any* other line OQ, and therefore is the least of all the lines drawn from O to AB.

Next let OQ, OR be two obliques of which OR is the further from the perpendicular, that is, makes the greater angle with it; then OR shall be greater than OQ.

Proof. For since OPQ is a right angle, OQR the exterior angle is an obtuse angle (Th. 2, Cor. 4); and ORQ is an acute angle; therefore OQR is $> ORQ$,

and therefore $\qquad OR$ is $> OQ$ (Theorem 11).

Lastly, there can be drawn one, and only one, oblique equal to a given oblique, which will be equally inclined to the perpendicular and on the opposite side of it.

Proof. For if $PS = PR$, and the figure be conceived as folded on OP, it is clear that since the angles at P are right

angles, PS would fall on PR, and S on R, and therefore OS on OR. Therefore $OS=OR$. Moreover the angle $POS=$ the angle POR, that is, OS and OR are equally inclined to the perpendicular OP on opposite sides of it.

Hence too it is evident that not more than two equal obliques can be drawn, for of the two on one side of the perpendicular one must be more remote from it than the other, and therefore must be greater than the other.

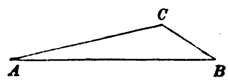
COR. 1. *Of two obliques, whether on the same or opposite sides of the perpendicular, the longer is more remote from the perpendicular than the shorter.*

COR. 2. *Hence the distance of a point from a line is the perpendicular let fall from the point on the line.*

THEOREM 13.

Any two sides of a triangle will be together greater than the third side.

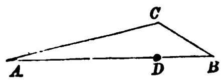
Proof. For in any triangle ACB, AB is the straight line joining A and B, and is therefore shorter than the broken line ACB, Ax. 1.



COR. *It follows that the difference of any two sides of a triangle will be less than the third side.*

For from AB cut off $AD=AC$.

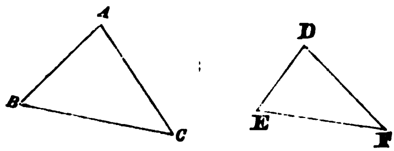
Then DB is the difference between AB and AC. Since therefore $AC+CB>AD+DB$, and $AC=AD$, it follows that CB is greater than DB; that is, DB is less than CB.



The next seven theorems contain properties of a pair of triangles.

THEOREM. 14.

If two angles of one triangle are equal respectively to two angles of another triangle, then shall the third angles of the triangles be equal.



Let the two angles B and C of the triangle ABC be respectively equal to E and F of the triangle DEF. Then shall the angle A be equal to the angle D.

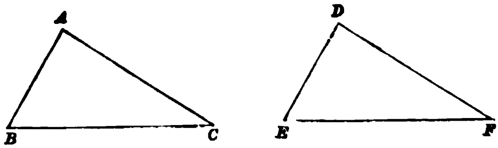
Proof. For A, B, and C together make up two right angles (Th. 7), and so also do D, E, and F. Therefore, since B and C are equal to E and F, it follows that A must be equal to D.

Remark. These triangles are said to be *equiangular to one another.*

NOTE.—By *corresponding* angles in any two triangles are meant those which are opposite to equal sides; and by *corresponding* sides are meant those which are opposite to equal angles.

THEOREM 15.

If two triangles are equiangular to one another and have a side of the one equal to the corresponding side of the other, then triangles will be equal in all respects.



Let ABC, DEF be the two triangles, in which the angles A, B, C are respectively equal to D, E, F; and one side BC equal to the corresponding side EF.

Then shall the triangles be equal in all respects.

Proof. For if the triangle ABC be conceived (Ax. 10) as placed on the triangle DEF, so that the side BC coincides with the equal side EF, then BA will fall on ED, since the angle B = the angle E; and CA will fall on FD, since the angle C = the angle F.

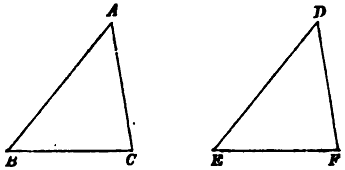
Therefore A will fall on D, and the triangles will coincide, and are therefore equal in all respects: that is, $AB=DE$, $AC=DF$, and the area of the triangle ABC = the area of the triangle DEF.

THEOREM 16.

If two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another, the triangles will be equal in all respects.

Let the two sides BA, AC of the triangle BAC be respectively equal to the two sides ED, DF of the triangle

EDF, and let them contain the angle $BAC =$ the angle EDF; then will the triangles be equal in all respects.

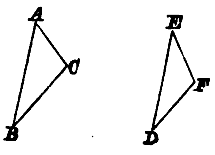


Proof. For the angle at A can be conceived (Ax. 10) as placed on the angle D, since these angles are equal, AB lying on DE, and AC on DF; and B and C would then coincide respectively with E and F, since $AB = DE$ and $AC = DF$.

Hence BC would fall on EF, since these lines would have two points in common (Ax. 2); and therefore BC must be equal to EF, and the angles B and C respectively coincide with and are equal to the angles E and F; and the area of the triangle ABC is equal to the area of the triangle DEF.

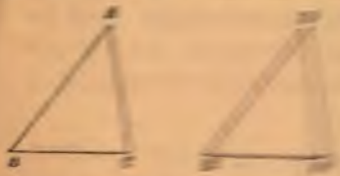
THEOREM 17. Oppositely.

If two sides of one triangle are respectively equal to two sides of another, but the included angles are unequal, the bases shall be unequal, that base being the greater which is opposite the greater angle.



Let the two sides BA, AC of the triangle BAC be respectively equal to the two sides DE, EF of the triangle

let them contain the angle [illegible]
will the triangles be equal in all respects.



[illegible] For the angle at [illegible]
the angle D, [illegible]
and AC on DF; [illegible]
ively with E and F, [illegible]
nce BC would fall on EF, [illegible]
wo points in common [illegible]
ual to EF, and the angles [illegible]
with and are equal to the angles [illegible]
of the triangle ABC is equal [illegible]
F.

Theorem [illegible]

If two sides of one triangle [illegible]
of another, [illegible]
shall be [illegible]
posite the greater [illegible]

[illegible] E,
[illegible] 6);
[illegible]

DF

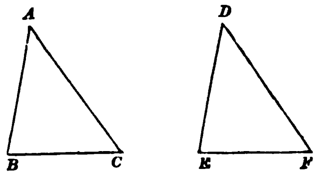
Theorem 20.

If two triangles have two sides of the one equal to two sides of the other, and the angle opposite that which is not the less of the two sides of the one equal to the corresponding angle of the other, the triangles shall be equal in all respects.

Let ABC, DEF be the two triangles, having the sides BA, AC equal to the sides ED, DF respectively, of which AC is not less than AB, and having also the angle $B=$ the angle E.

Then shall the triangles be equal in all respects.

Proof. For if $AC=AB$, and $DE=DF$, the angles B, C, E, F are equal, and the remaining angle $A=$ the remaining angle D, and the triangles are equal in all respects by Theorems (15) or (16).

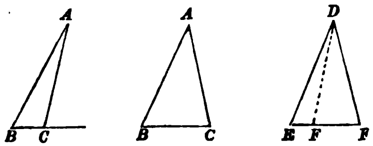


If AC is $> AB$, then since $AB=DE$ it could be placed so as to coincide with it; and since the angle $B=$ the angle E, the line BC would coincide in direction with EF, and the point C would fall somewhere on EF, or on EF produced through F.

It remains to see the effect of the only remaining condition that $AC=DF$.

Now the line AC has one extremity A on D, and the other extremity C somewhere on EF, or on EF produced through F, and $AC = DF$. But there is no oblique equal to DF, and not coinciding with it, that can be drawn from D to any point in EF; for since DF is $> DE$ it is more remote from the perpendicular (Th. 12, Cor.); and therefore the oblique equal to DF would lie on the other side of DE, and have its extremity in EF produced through E.

Hence AC would coincide with DF, and the triangles are therefore equal in all respects.



COR. 1. *If the side opposite the given angle were less than the side adjacent, there would be two triangles, as in the figure; and the proof given above is inapplicable.*

This is called *the ambiguous case.*

COR. 2. *If the given angle is a right angle, the side opposite to it must be greater than the side adjacent; by Th. 11. Hence if two right-angled triangles have the hypothenuse and one side of the one equal respectively to the hypothenuse and one side of the other, the triangles are equal in all respects.*

This corollary is of very frequent use.

COR. 3. *A similar property is obviously true of two obtuse-angled triangles.*

Exercises on Triangles.

The theorems respecting triangles, which follow from the elementary properties of triangles proved above, are of two kinds; which announce respectively the equality, or the inequality of lines and angles.

Theorems of equality depend on Theorems 7, 8, 10, 14, 15, 16, 18, 20, and theorems of inequality on Theorems 9, 11, 12, 13, 17, 19, given above.

We shall give a few examples of these theorems with demonstrations, and add some exercises which may be left to the student's ingenuity to prove. They require an application of these general theorems to the special data of the theorem proposed.

The general method to be adopted in the solution of theorems of equality is the following. Examine fully the statement of the question; see what is included among the *data:* what lines and angles are *given* equal. Then see what is required to be proved, what lines or angles have to be proved to be equal. It may follow from the properties proved (in Theorems 7, 8, 10) of a single triangle; or it may depend on the equality of a pair of triangles. Examine the triangles of which they form corresponding parts, and see whether the data are sufficient to prove *these* triangles equal. If the data are sufficient, the solution is effected by comparing the triangles, and shewing the required equality of the lines and angles; if not, the data must be used to establish results, which in their turn can be used to establish the conclusion required.

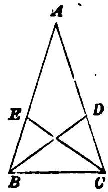
The beginner will do well to arrange his proofs in the manner shewn in the examples, giving references in the margin.

Theorems of Equality.

1. *The lines which bisect the angles at the base of an isosceles triangle, and meet the opposite sides, are equal.*

Let ABC be an isosceles triangle.

Data. $AB = AC$, and the angles at B and C bisected by BD, CE.



Proof. In the triangles AEC, ADB

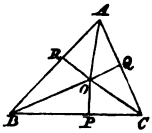
$$\left.\begin{array}{ll} \text{we have} \quad AC = AB. & \textit{Data.} \\ \text{angle at } A \text{ common} & \\ \text{and angle } ACE = \text{angle } ABD: & \textit{Data \& Th. 8.} \end{array}\right\}$$

Therefore the base CE = the base BD, Th. 15.

2. *The bisectors of the three angles of a triangle meet in one point.*

Let ABC be a triangle and let the bisectors of the angles ABC, ACB be BO, CO, meeting in O; then the theorem will be proved if we can shew that AO is the bisector of the angle BAC.

Let perpendiculars OP, OQ, OR be drawn to the three sides BC, CA, AB.



Proof. In the triangles OQC, OPC

$$\text{we have} \quad \left.\begin{array}{ll} OQC = OPC & \textit{Const.} \\ OCQ = OCP & \textit{Data.} \\ OC \text{ common.} & \end{array}\right\}.$$

Therefore $OQ = OP$ by Theorem 15.

Similarly from the triangles OPB, ORB, it follows that $OP = OR$; therefore $OR = OQ$;

And therefore the right-angled triangles OQA, ORA have the hypothenuse and one side of the one equal to the hypothenuse and one side of the other, and are therefore equal in all respects by Theorem 20, Cor. 2.

Therefore the angle OAQ = the angle OAR, that is, OA is the bisector of the angle BAC.

Exercises on Triangles.

Theorems of Equality.

1. OA and OB are any two equal lines, and AB is joined; shew that AB makes equal angles with OA and OB.

2. If the bisectors of the equal angles B, C, of an isosceles triangle meet in O, shew that OBC is also an isosceles triangle.

3. If ABC is an isosceles triangle and A is double of either B or C, shew that A is a right angle.

4. If ABC is an isosceles triangle and A is half of either B or C, shew that A is two-fifths of a right angle.

5. Find the angle between the lines that bisect the angles at the base of the triangle in the last question.

6. The perpendiculars let fall from the extremities of the base of an isosceles triangle on the opposite sides will include an angle supplementary to the vertical angle of the triangle.

7. The line drawn to bisect the vertical angle of an isosceles triangle will also bisect the base, and be perpendicular to it.

8. The lines joining the middle points of the sides of an isosceles triangle to the opposite extremities of the base will be equal to one another.

9. The line drawn from the vertex of an isosceles triangle to bisect the base, will cut it at right angles, and bisect the vertical angle.

10. The perpendiculars let fall from the extremities of the base of an isosceles triangle upon the opposite sides will be equal, and will make equal angles with the base.

11. The perpendicular let fall from the vertex of an isosceles triangle to the base, will bisect the base and the vertical angle.

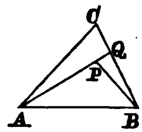
12. If two exterior angles of a triangle be bisected by straight lines which meet in O, prove that the perpendiculars from O on the sides or sides produced of the triangle are equal to one another.

13. Prove that the lines which bisect the sides of a triangle and are perpendicular to them meet in one point.

Theorems of Inequality.

1. *If* P *is any point within the triangle* ABC, *prove that* PA + PB < CA + CB, *but the angle* APB > *the angle* ACB.

Proof. Produce AP to meet BC in Q.



Then $AC+CQ>AQ$ (Th. 13),

and therefore $AC+CB>AQ+QB$;

also $PQ+QB>PB$ (Th. 13),

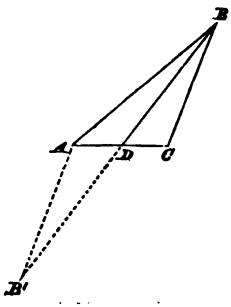
and therefore $AQ+QB>AP+PB$;

much more then are $AC+CB>AP+PB$.

And the angle $APB>AQB$, and $AQB>ACB$ (Th. 7, Cor.); much more then is $APB>ACB$.

2. *The line that joins the vertex to the middle point of the base of a triangle is less than half the sum of the two sides.*

Let D be the middle point of AC, then is BD less than half the sum of AB, BC.



Proof. Produce BD to B', making $DB'=DB$. Join AB'.

Then since the two triangles BDC, ADB' have two sides BD, DC and the included angle BDC of the one respectively equal to the two sides $B'D$, DA

and the included angle $B'DA$ of the other, therefore (Theorem 16) the base BC = the base AB';

but $$B'A + AB > B'B \text{ (Th. 13)},$$

$$\therefore AB + BC > B'B, \text{ which is twice } BD,$$

that is, BD is less than half the sum of AB and BC.

The line BD is called a *median* of the triangle.

Exercises on Triangles.

Theorems of Inequality.

1. Prove that any one side of a four-sided figure is less than the sum of the other three sides.

2. Prove that the sum of the lines which join the opposite angles of any four-sided figure is together greater than the sum of either pair of opposite sides of the figure.

3. Prove that the sum of the diagonals of a quadrilateral figure is less than the sum of the four lines which can be drawn to the angles from any other point than the intersection of the diagonals.

4. O is any point within the triangle ABC; prove that $OA + OB + OC$ are less than the sum, and greater than half the sum of $AB + BC + CA$.

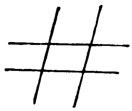
5. Prove that the sum of the four sides of a quadrilateral figure is greater than the sum and less than twice the sum of the diagonals.

6. If ABC is a triangle in which AB is greater than AC, and D is the middle point of BC, and AD is joined, prove that the angle ADB is an obtuse angle.

7. Prove that the sum of the three sides of a triangle is greater than the sum of the three medians.

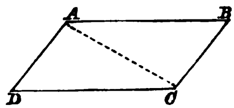
SECTION IV. PARALLELOGRAMS.

Def. 30. A four-sided figure of which the opposite sides are parallel is called a *parallelogram.*



THEOREM 21.

The opposite angles and sides of a parallelogram will be equal, and the diagonal, or the line which joins its opposite angles, will bisect it.



For, first, since the lines which meet in B are respectively parallel to the lines which meet in D, and are both drawn in opposite directions from those points, the angle at $B =$ the angle at D by Theorem 4.

Similarly the angle at $A =$ the angle at C.

Again, if AC be joined, since the angles BAC, BCA are respectively equal to their alternate angles DCA, DAC, and AC is common, the two triangles ABC, CDA will be equal in all respects (by Theorem 15), and therefore $AB = CD$, $BC = AD$, and the area $ABC =$ the area ADC.

COR. 1. *It is obvious that the triangle* ADC *is half the parallelogram* ADCB.

COR. 2. *If one of the angles of a parallelogram is a right angle, the others also are right angles.*

Def. 31. A parallelogram which is right-angled is called a *rectangle.*

Def. 32. A parallelogram all whose sides are equal is called a *rhombus.*

Def. 33. A rectangle which has all its sides equal is called a *square.*

Def. 34. A four-sided figure which has only one pair of opposite sides parallel is called a *trapezium.*

EXERCISES ON PARALLELOGRAMS.

1. Shew that a trapezium may be divided into a parallelogram and a triangle.

2. Or into a rectangle and two right-angled triangles.

3. The diagonals of any parallelogram will bisect one another.

4. The diagonals of a rhombus will bisect one another at right angles.

5. If two straight lines be drawn bisecting one another, and their extremities be joined, the figure so formed will be a parallelogram.

6. Given that a four-sided figure has its opposite sides equal, prove that it must be a parallelogram.

7. Prove that the diagonals of a rectangle are equal to one another.

8. The straight lines which join the extremities of equal and parallel straight lines towards the same parts will be themselves also equal and parallel.

9. Prove that parallel straight lines are everywhere equidistant.

SECTION V. PROBLEMS OF CONSTRUCTION.

In the Science of Geometry there are not only theorems to be proved, but constructions to be effected, which are called *problems*. Geometers have always imposed certain limitations on themselves with respect to the instruments which might be used in these constructions. There is no reason why any convenient instrument used in the Art should not be supposed to be used in the Science of Geometry, such as the square, parallel ruler, sector, protractor; but the ruler and compasses suffice for nearly all the simpler constructions, and those which cannot be effected by their means are considered as not forming a part of Elementary Geometry. There are some problems, that seem at first sight not very difficult, that cannot be solved by the use of these instruments. We can, for example, bisect an angle; but we cannot, in general, trisect it, that is, divide it into three equal parts, by any combination of ruler and compasses. It may be observed that the ruler is simply a straight edge, not graduated, and the compasses are supposed to be transferable from one part of the figure to another, the distance between the points being unaltered.

The solution of a problem in Elementary Geometry as above defined consists

(1) in indicating how the ruler and compasses are to be used in effecting the construction required;

(2) in proving that the construction so given is correct;

(3) in discussing the limitations, which sometimes exist, within which alone the solution is possible.

We shall give several examples of such problems, and then discuss the principles of the methods we have used. It will be observed that in the figures we make the given lines thick, the resulting line or lines thin, and the lines used only in the construction or proof dotted.

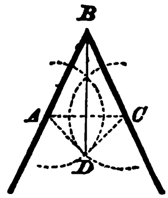
Def. 35. *A circle* is a plane figure contained by a line called the *circumference* such that all the points in that line are equally distant from a certain point which is called the *centre* of the circle. The distance of any point on the circumference from the centre is called the *radius* of the circle.

PROBLEM I.

To bisect a given angle, that is, to divide it into two equal parts.

Construction. Let ABC be the given angle.

Take any equal lengths BA, BC, and join AC.



With centre A and any radius greater than half AC describe a circle, and with centre C and the same radius describe another circle intersecting the former circle in D.

Join AD, CD, and BD;

Then BD shall bisect the angle ABC.

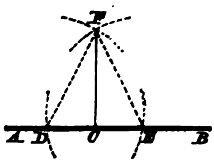
Proof. For the triangles ABD, CBD have obviously the three sides of the one equal (by the construction) to the three sides of the other, and therefore (by Theorem 18) the corresponding angles ABD, CBD are equal.

Remark. It is assumed here that if a circle has one point inside another circle, the circumferences will intersect one another.

PROBLEM 2.

To draw a perpendicular to a given straight line from a given point in it.

This is a particular case of Problem 1. For let AB be the given line, C the given point in it; then ACB is a straight angle, which it is required to bisect. The construction is therefore the same.



Construction. With centre C and any radius describe a circle to cut the straight line in two points D, E, so that $CD = CE$.

With centre D and any radius greater than DC describe a circle, and with centre E and the same radius describe a circle, cutting the former in F.

Join FC.

Then FC is perpendicular to AB.

Proof. For the triangles DCF, ECF have by the construction the three sides of the one equal respectively to the three sides of the other.

And therefore the corresponding angles DCF, ECF are equal to one another (Th. 18), and therefore they are right angles by Def. 10.

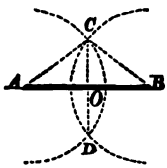
NOTE.—This construction is usually effected in practice by means of the square.

PROBLEM 3.

To bisect a given straight line.

Let AB be the given straight line.

Construction. With centre A and any radius greater than half AB describe a circle, and with centre B and the same radius describe a circle intersecting the former in two points C and D.



Join CD cutting AB in O.

Then O will be the point of bisection.

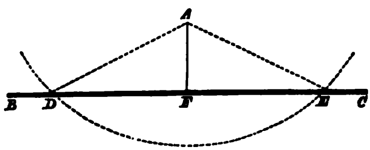
Proof. For by the last Problem CD is so drawn as to bisect the angle ACB;

And then in the triangles ACO, BCO we have $AC = BC$, CO common, and the included angles ACO, BCO equal:

And therefore by Theorem 16, the base $AO =$ the base BO: that is, O is the point of bisection of AB.

PROBLEM 4.

To draw a perpendicular to a given straight line from a given point without it.



Let BC be the given straight line, A the given point.

Construction. With centre A describe a circle with any sufficient radius to cut BC in two points D, E.

Bisect DE in F (Prob. 3). Join AF.

Then AF shall be perpendicular to BC.

Proof. For if AD, AE be joined, it is clear that the triangles AFD, AFE have the three sides of the one respectively equal to the three sides of the other.

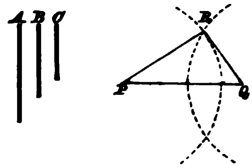
Therefore the angle AFD = the corresponding angle AFE (Th. 16), and therefore AF is perpendicular to BC (Def. 10).

NOTE.—This construction also is usually effected in practice by means of the square.

A large class of problems consists of those in which a triangle has to be constructed under certain conditions. We shall take the most important cases, which are also the simplest.

PROBLEM 5.

To construct a triangle, having given the lengths of the three sides.



Let the three given lengths be the lines A, B, C.

Construction. Draw a line PQ equal to one of them A. With centre P and radius equal to B describe a circle; and with centre Q and radius equal to C describe a circle. Let these circles intersect in R. Join RP, RQ.

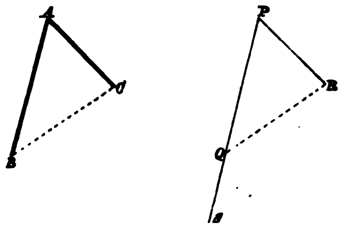
RPQ is the triangle required.

Proof. For its three sides are by the construction equal to the three given lines.

Limitation.—It is necessary that any two of the lines A, B, C should be together greater than the third. For if B and C were together less than A, the circles in the figure would obviously not meet: and if they were together equal to A, the point R would be on PQ, and the triangle would be-

come a straight line. Similarly if B were greater than $A+C$ or C greater than $A+B$, the circles would not intersect. This limitation might be anticipated from the theorem before proved, that any two sides of a triangle are together greater than the third side.

Cor. 1. Hence it is possible with the ruler and compasses alone to effect what is performed in practice generally by means of the sector or protractor, viz. *to make an angle equal to a given angle.*



Let BAC be the given angle.

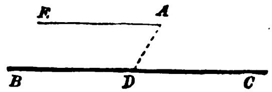
Join any two points B, C in its sides. Construct a triangle PQR having its three sides PQ, QR, RP respectively equal to AB, BC, CA.

Then (by Theorem 18) the angle P is equal to the angle A.

Cor. 2. *An angle equal to a given angle may be made at any point, and such that one of the lines containing it may be any given line through the point.*

Thus if P were the given point, PS the given line, PQ must be taken equal to AB, and the rest of the construction is the same as before.

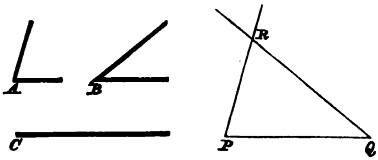
COR. 3. *Hence through any point a straight line can be drawn parallel to a given straight line.*



Let A be the given point, BC the given line. Draw any line AD to meet BC, and make the angle DAE equal to the alternate angle ADC. Then by Theorem 5, Cor. 2, AE is parallel to BC.

PROBLEM 6.

To construct a triangle, having given two angles and a side adjacent to both.



Let A, B be the two angles, C the given side.

Take a line $PQ = C$. At the points P, Q make angles equal respectively to A and B (Prob. 5, Cor. 1). Let the lines which contain these angles meet in R.

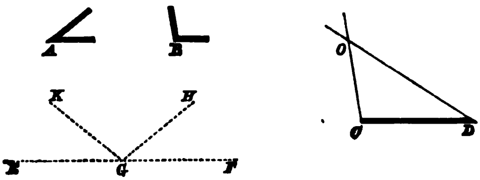
Then RPQ is the triangle required.

Limitation.—The two given angles must be together less than two right angles, or the lines PQ, QR would not meet. This follows also from the theorem that the three interior angles of a triangle are together equal to two right angles.

PROBLEM 7.

Having given two angles and a side opposite to one of them, to construct the triangle.

Let A and B be the given angles, CD the given side which is to be opposite to A.



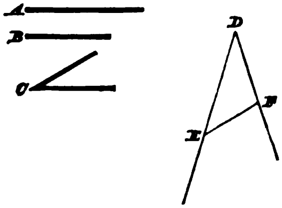
Construction. Draw an indefinite straight line EF. At any point G in it make the angles $FGH=A$, and $HGK=B$ (Prob. 5), then KGE will equal the third angle of the triangle, since the sum of the three angles of a triangle is equal to two right angles (Th. 7). At C and D make angles equal to HGK and KGE, and let their sides meet in O; then OCD is the triangle required.

Proof. For the angle O is the supplement of the angles $OCD+ODC$, and must therefore be equal to HGF, that is, to A.

Limitation.—As before the two given angles must be together less than two right angles.

PROBLEM 8.

To construct a triangle, having given two sides and the angle between them.



Let A, B be the given sides, C the given angle.

Construction. Draw an angle D equal to the given angle, and take DE, DF equal to A and B. Join EF.

This problem needs no proof.

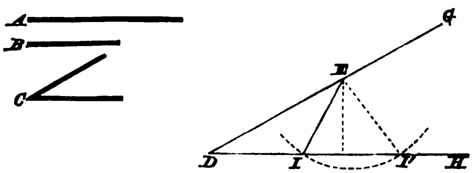
Remark. In these problems we have found that one triangle and only one can be constructed to fulfil the conditions given. In other words that with these *data* the triangle is *determinate.* Also we notice that in each case *three* elements in the triangle are *data* or given. We have given either the three sides, or two angles and the side adjacent to both, or two angles and a side opposite to one, or two sides and the included angle. And these cases correspond to the theorems proved above of the equality of triangles. For if *only one* triangle can be constructed so as to have its sides equal to three given lines, it is clear that if two triangles have the three sides of the one equal to the three sides of the other, these triangles must be identical, or be equal in all respects. And a similar remark may be made on the other cases we have considered.

But there are cases in which the data may be insufficient to determine the triangle. For example, if only two sides are given, an indefinite number of different triangles may be constructed to have these sides. Or if the three angles are given, their sum being equal to two right angles, an indefinite number of triangles may be constructed to have these three angles. And again it may be impossible to construct the triangle with the given data, as has been already shewn. In some cases moreover the solution is *ambiguous, that is, there may be more than one triangle* which fulfils the given conditions. The following is an important instance of this, and is usually called *the ambiguous case*, spoken of in Theorem 20.

PROBLEM 9.

To construct a triangle having given two sides and an angle opposite to one of them.

Let A, B be the given sides, C the angle to be opposite to the side B.



Take an angle $GDH = C$, take $DE = A$, and with centre E and radius $= B$ describe a circle. If I is one of the points in which this circle meets the line DH, by joining EI we obtain a triangle which fulfils the given conditions.

But several cases may arise.

Let the given angle be acute, as in the figure.

Then, by Theorem 12,

(1) If B is less than the perpendicular from E on DH, the circle would not meet DH, and the triangle would be *impossible.*

(2) If B is equal to the perpendicular, the circle would meet DH at the foot of the perpendicular, and there would be *one triangle, right-angled,* which fulfils the given conditions.

(3) If B is greater than the perpendicular but less than DE, then the circle will meet DH in two points I, I' as in the figure, on the same side of D, and there will be *two triangles* EDI, EDI' which fulfil the given conditions.

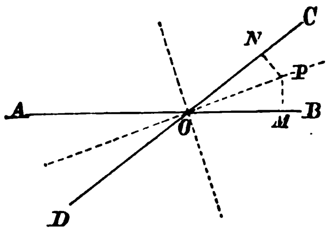
(4) If B is greater than DE, the circle will meet DH in two points on the opposite sides of D, but one only of the triangles made by joining EI, EI' will be found to have the angle D, and the other will have the supplementary angle: that is, there will be only *one solution.*

The cases of the given angle being a right angle or an obtuse angle we leave to the ingenuity of the student.

Loci.

When a point is required which is to satisfy one geometrical condition, the problem is generally indeterminate, that is, there is an infinite number of points which satisfy the condition. For example, if a point is to be found equally distant from two given straight lines of indefinite

length AOB, COD, it is clear that the condition is satisfied by any number of points like those marked in the figure.



The assemblage of such points as fulfil the given condition is called the locus of these points. In the example given, the locus consists of the bisectors of the angles at O, which form two straight lines at right angles: and the proof consists in shewing that if P be any point on the bisector OP, and PN, PM are perpendiculars on OB, OA; the triangles PON, POM are equiangular to one another, and have a side OP common, and therefore $PN = PM$ (Th. 15). Therefore every point in both bisectors is equidistant from the given lines.

Exercises.

Find the following loci:—

(1) Of a point at a given distance from a given point.

(2) Of a point at a given distance from a given line.

(3) Of a point at a given distance from a given circle.

(4) A horse is tethered by a chain fastened to a ring which slides on a rod bent into the form of a rectangle. Find the outline of the area over which he can graze.

(5) Find the locus of a point equidistant from two given points.

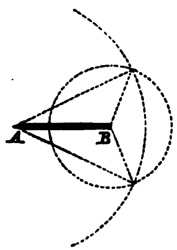
(6) Find the locus of points at which two equal lengths, adjacent or not adjacent, of a straight line subtend equal angles.

(7) Find part of the locus of points at which two adjacent sides of a square subtend equal angles.

INTERSECTION OF LOCI.

When a point has to be found which satisfies *two* conditions, the problem is generally determinate if it is possible: and the method of loci is very frequently employed in discovering the point. For if the locus of points which satisfy each condition separately is constructed, it is obvious that the points which satisfy both conditions must be the points common to both loci, that is, must be the point or points where the loci intersect.

For example; a triangle is to be constructed on a given base with its sides of given lengths. Let AB be the base.



The two conditions are that the lengths of the two sides are given; the point sought for is the vertex: now the vertex must be at a certain distance from A = one of the given lengths; its locus is therefore a certain circle round A as centre. Similarly it must be at a certain distance from B; its locus is therefore another circle round B as centre. The points of intersection of these circles are therefore the vertices of the two equal triangles which fulfil the given conditions.

It was this reasoning that suggested the construction in Problem 5.

Occasionally it will be found that with certain data in the following exercises the loci do not intersect, or the solution becomes impossible. As in the case given, it will not be difficult to see that the circles would not intersect unless any two of the sides were greater than the third side.

EXERCISES ON INTERSECTION OF LOCI.

1. Find a point in a given straight line at equal distances from two given points. Construct the figures for all cases.

2. Find a point in a given straight line at a given distance from a given straight line.

3. Find a point in a given straight line at equal distances from two other straight lines.

4. On a given straight line to describe an equilateral triangle.

5. Describe an isosceles triangle on a given base, each of whose sides shall be double of the base.

6. Find a point at a given distance from a given point, and at the same distance from a given straight line.

7. Given base, sum of sides, and one of the angles at the base, construct the triangle.

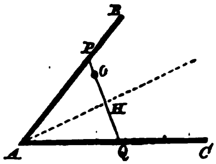
8. Given base, difference of sides, and one of the angles at the base, construct the triangle.

9. Find a point at a given distance from the circumference of two given circles, the distances being measured along their radii or their radii produced.

Analysis and Synthesis.

If problems cannot be solved by this method, it remains to attack them by the method, as it is called, of Analysis and Synthesis. This is not so much a method as a way of searching for a suggestion, and nothing but experience and ingenuity will here avail the student. The solution is supposed to be effected, and relations among the parts of the figure are then traced until some relation is discovered which can give a clue to the construction. Nothing but seeing examples can make this clear.

(1) *It is required to draw a line to pass through a given point and make equal angles with two given intersecting lines.*



Let O be the given point, AB, AC the given lines.

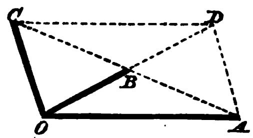
We reason as follows (*analysis*): suppose POQ were the line required, then the angle at P = angle at Q.

Therefore $AP = AQ$; therefore if we bisected the angle A, POQ would be at right angles to the bisector.

Now this is a suggestion we can work backwards from, and the construction is

Synthesis. Bisect the angle BAC, and let fall OH a perpendicular to the bisector, and let it meet the lines in P, Q, and POQ can then be proved to be the line required.

(2) *It is required to draw from a given point three straight lines of given lengths, so that their extremities may be in the same straight line, and intercept equal distances on that line.*



Analysis. Suppose OA, OB, OC were the three lines, so that CBA is a straight line, and $CB = BA$.

Then it occurs to us that if OB were prolonged to D, making $BD = OB$, then CD and DA would be respectively parallel and equal to OA and OC (see § 4, Ex. 5); and that the sides of the triangle DOA are respectively equal to OA, OC and $2OB$. Hence the construction is suggested.

Synthesis. Make a triangle DOA whose sides are OA, OC, and $2OB$; complete the figure, by drawing DC, OC, parallel to OA, AD; and the other diagonal ABC will be the line required. For it may be shewn that $AB = BC$.

The student must not be surprised if he finds problems of this class difficult. For there is nothing to point out which of the many relations of the parts of the figure are to be followed up in order to arrive at the particular relation which suggests the construction. It is not easy to see what is to suggest the producing of OB to D as in the figure.

Subjoined are a few problems of no great difficulty, which may be solved by this method.

Problems.

1. On a given straight line to describe a square.

2. Describe a rectangle with given sides.

3. Given two sides of a parallelogram and the included angle, construct the parallelogram.

4. Given the lengths of the two diagonals of a rhombus, construct it.

5. From a given point without a given straight line to draw a line making an angle with the line equal to a given angle.

6. Describe a square on a given straight line as diagonal.

7. Draw through a given point between two straight lines not parallel, a straight line which shall be bisected in that point.

8. Place a line of given length between two intersecting lines so as to be parallel to another given line.

9. Trisect a right angle.

10. Divide half a right angle into six equal parts.

11. Three straight lines meet in a point, draw a straight line such that the parts of it intercepted by the three lines shall be equal to one another.

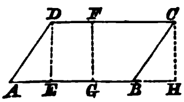
12. Trisect a given straight line.

SECTION VI. THE EQUIVALENCE OF FIGURES.

By *equivalent figures* are meant figures whose areas are equal although the figures may be of different shapes, and therefore not conceivable as superposed on one another. Thus a circular field may be as large as a square one, or a triangular piece of paper as large as a rectangular piece, and in such cases these figures would be called *equivalent*. The consideration of equivalent figures is an important part of Geometry.

Def. 36. The *altitude* of a parallelogram is the perpendicular distance between one side which is called the base and the side parallel to it.

Thus in the figure at the side the perpendiculars *DE*, *FG*, or *CH*, which are equal (by Theorem 21) since *DEGF*, *DEHC* are parallelograms, are each of them the altitude of the parallelogram *ABCD*, *AB* being the base.

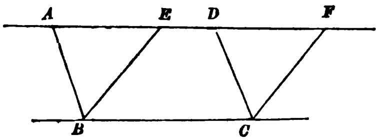


Any side of a triangle being taken as base, the altitude of the triangle is the perpendicular let fall on that side, or that side produced, from the opposite angle.

Theorem 22.

Parallelograms on the same base and between the same parallels will be equivalent.

Let $ABCD$, $EBCF$ be parallelograms on the same base BC, and between the same parallels AF, BC. They shall be equivalent.



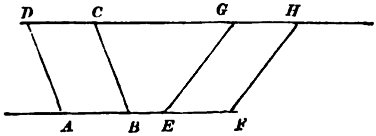
Proof. For in the triangles BAE, CDF the angle A = the angle D, and the angle E = the angle F, by parallelism (Th. 4), and the side AB = the corresponding side DC, and therefore the triangles are equal in area, by Theorem 15.

But if the triangle CDF is taken away from the trapezium $ABCF$, the parallelogram $ABCD$ remains: and if the triangle BAE is taken away from the same trapezium the parallelogram $EBCF$ remains. Therefore these parallelograms are equivalent.

Remark. This theorem is the fundamental theorem of equivalence of areas.

COR. 1. *A parallelogram is therefore equivalent to the rectangle on the same base with the same altitude, and its area is therefore* determinate *when its base and altitude are known.*

COR. 2. *Parallelograms on* equal *bases and between the same parallels are equivalent.*



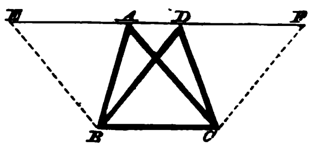
Proof. For if the base AB = the base EF, the parallelogram $AEFG$ can be conceived as superposed on the parallelogram $ABCD$, EF coinciding with AB, so that they should have the same base, and be between the same parallels, and therefore they are equivalent.

COR. 3. *Parallelograms on equal bases and of the same altitude are equivalent.*

For similarly they may be conceived as placed on one another so as to have the same base, since the bases are equal, and so as to be between the same parallels, since the altitudes are equal; and therefore they are equivalent.

THEOREM 23.

Triangles on the same base and between the same parallels will be equivalent.



Let ABC, DBC be triangles on the same base BC and between the same parallels, they shall be equivalent.

Proof. For if BE be drawn from B parallel to AC, and CF be drawn from C parallel to BD, to meet AD produced in E and F,

Then $ACBE$, $DBCF$ are parallelograms, and are equivalent to one another (by Theorem 22).

But the triangles ABC, DBC are respectively the halves of the parallelograms $ACBE$, $DBCF$ (Th. 21).

Therefore the triangles are equivalent.

COR. 1. *Triangles on equal bases and between the same parallels are equivalent.*

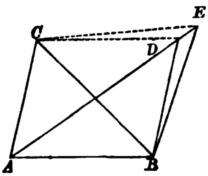
Cor. 2. *Triangles on equal bases and of the same altitude are equivalent. Hence a triangle is of determinate area when its base and altitude are given.*

Cor. 3. *If a parallelogram and a triangle have equal bases and altitudes, the parallelogram is double of the triangle.*

Proof. For the parallelogram $ACBE$ is double of the triangle ABC, and therefore of any other triangle DBC which is on the same base, and between the same parallels.

Theorem 24. *Conversely.*

Equivalent triangles on equal bases will be of equal altitude.



For let the triangles ABC, ABD be equivalent triangles on equal bases, and let them be conceived as placed on the same base AB, and on the same side of it.

Then shall their altitudes be equal, that is, CD will be parallel to AB.

Proof. For if CD were not parallel to AB, suppose that some other line through C, as CE, meeting AD in E, were parallel to AB. Then the triangles CAB, EAB would be equivalent by Theorem 23.

But we know that CAB, DAB are equivalent, therefore EAB and DAB would be equivalent, which is absurd.

Cor. *Hence equivalent triangles on the same base, and on the same side of it, must be between the same parallels.*

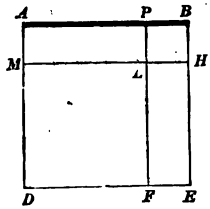
Def. 37. A rectangle is determined when two of its adjacent sides are known. It is then said to be *contained by* its two adjacent sides, or by lines equal to them.

Def. 38. A line AB is said to be *internally* divided in P when P lies between A and B: and it is said to be *externally* divided in P when P lies in AB produced: and AP, BP are called in both cases the *segments* of AB.

Theorem 25.

If a straight line is divided internally in any point, the square on the line will be equal to the squares on the two segments together with twice the rectangle contained by the segments.

Let AB be divided internally in P.



Then will the square on AB be equal to the squares on AP, PB together with twice the rectangle contained by AP, PB.

Proof. Describe a square $ADEB$ on AB.

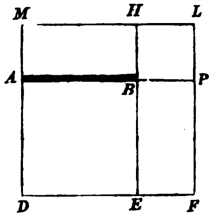
Through P draw PLF parallel to AD, meeting DE in F: cut off $PL = PB$. Through L draw HLM parallel to AB, to meet DA and EB in M, H.

Then it may be seen that the figures AL, PH, LE, MF are parallelograms by construction; and it is easily shewn that PH, MF are the squares on PB, AP respectively; and that AL, LE are each of them the rectangle contained by AP, PB.

Hence, since $ADEB$ is made up of these four figures, it follows that the square on AB is equal to the squares on AP, PB and twice the rectangle contained by AP, PB.

Theorem 26.

If a straight line be divided externally in any point, the square on the line is equal to the squares on the two segments diminished by twice the rectangle contained by the segments.



Let AB be divided externally in P.

Proof. Describe a square $ADEB$ on AB.

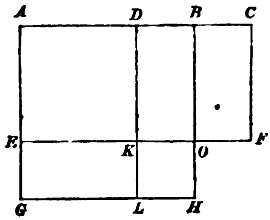
Through P draw LPF parallel to AD, meeting DE produced in F: cut off $PL = PB$. Through L draw MHL parallel to AB, to meet DA, and EB produced in M, H.

Then as before it is evident that MF is the square on AP; and MP or HF, the rectangle contained by AP and BP; and HP the square on BP.

And AE is less than $MF + HP$ by $MP + HF$; that is, the square on AB is equal to the squares of AP, PB diminished by twice the rectangle contained by AP, PB.

THEOREM 27.

The rectangle contained by the sum and difference of any two lines is equivalent to the difference of the squares on those lines.



Let AB, BC be the two lines, then AC is the sum of the lines, and if BD be taken equal to BC, AD is their difference.

Proof. Draw $AE = AB$ at right angles to AC, and complete the parallelogram $AEFC$, which is therefore the rectangle contained by AC, AD, that is, by the sum and difference of AB, BC.

On AB describe the square $AGHB$; through D draw DKL parallel to AG: then KH is the square on BC.

Then it may be seen that EL, DO, BF are equal figures; but the difference of the squares on AB, BC is the figure made up of AO and EL, that is, it is equivalent to the figure made up of AO and BF, that is, to AF, which is the rectangle contained by the sum and difference of AB, BC.

COR. *If a straight line is bisected and divided in any point, the rectangle contained by the segments is equal to the difference of the squares on half the line and the line between the points of section.*

Proof. For let AB be bisected in C, and divided internally or externally in P.

Then AP is the sum of AC and CP, and PB is their difference, since $BC = AC$.

Therefore the rectangle contained by AP, PB is equal to the difference of the squares of AC, and CP.

Remark. The student will begin here to suspect, what he will afterwards find to be true, that there is an intimate relation between geometry and algebra. Algebraical or analytical geometry as it is called, investigates this relation and applies it to the establishment of theorems in geometry, and will occupy him at a later stage of his mathematical studies. We shall at present use the expression AB^2, which is read 'AB squared,' only as an *abbreviation* for "the square on AB," and $AB \times AC$ or $AB \, . \, AC$, as an abbreviation for "the rectangle contained by AB and AC."

These three theorems may be used to demonstrate other properties of divided lines. For example,

THEOREM 28.

If a straight line be divided into two equal and also into two unequal segments, the squares of the two unequal segments are together double of the square of half the line bisected, and the square on the line between the points of section.

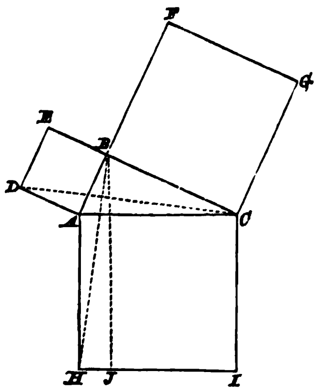
Let AB be bisected in C, and divided internally or externally in D.

Then the squares on AD, DB will be double of the squares on AC, CD.

Proof. For $AD^2 = AC^2 + CD^2 + 2AC \times CD$ by Th. 25;
and $DB^2 = CB^2 + CD^2 - 2BC \times CD$ by Th. 26;
therefore, adding, and remembering that $AC = BC$, and that therefore the rectangle $AC \times CD$ = the rectangle $BC \times CD$, we get that $AD^2 + DB^2 = 2AC^2 + 2CD^2$.

THEOREM 29.

In any right-angled triangle the square on the hypothenuse is equivalent to the sum of the squares on the sides which contain the right angle.



Let ABC be a triangle right-angled at B. Then will $AC^2 = AB^2 + BC^2$.

Proof. On AB, BC, CA describe the squares $ADEB$, $BFGC$, $CIHA$ respectively. Join CD, BH; and draw BJ parallel to AH.

Since the angles ABC, ABE, CBF are right angles, it follows that CBE, ABF are straight lines (Th. 1).

Therefore the triangle DAC is on the same base DA, and between the same parallels DA, EC with the square $DABE$.

Therefore the triangle DAC is half the square $DABE$ (Th. 23, Cor. 3).

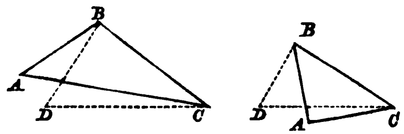
Similarly the triangle BAH is half the rectangle AJ.

But the triangles DAC, BAH are equal (Th. 16); for the sides DA, AC are respectively equal to BA, AH, and the contained angle $DAC =$ the contained angle BAH, each of them being a right angle together with BAC.

Therefore the rectangle $AJ =$ the square $DABE$.

Similarly it may be shewn that the rectangle $CJ =$ the square $BCGF$, and therefore, since AJ and CJ make up the whole square $AHIC$, the square $AHIC$ is equivalent to the sum of the squares $ABDE$ and $BCGF$, that is, $AC^2 = AB^2 + BC^2$.

COR. 1. *The square of a side subtending an obtuse or acute angle is not equal to the sum of the squares of the side containing that angle.*



For if BD is drawn at right angles to BC and equal to BA, and DC joined, then AC is greater or less than DC, according as the angle CBA is obtuse or acute by Th. 17. Therefore AC^2 is greater or less than DC^2, that is, than $DB^2 + BC^2$ or than $AB^2 + BC^2$.

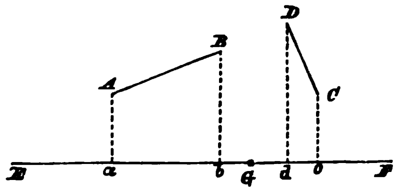
COR. 2. *Hence it follows that the converse theorem holds, viz. that if the square on one side of a triangle is equal to the squares on the other two sides, the triangle is right-angled.*

COR. 3. *It follows that in a triangle right-angled at B,*

$$AB^2 = AC^2 - BC^2 \text{ and } BC^2 = AC^2 - AB^2.$$

Def. 39. *The projection of one line on another line* is the portion of the latter intercepted between perpendiculars let fall on it from the extremities of the former.

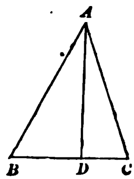
Thus the projections of AB, CD on EF are the lines ab, cd respectively.



It is clear that the line EF must be supposed indefinitely long. There could be no projection of AB on the terminated line GF.

THEOREM 30.

In any triangle the square on a side opposite an acute angle is less than the squares on the sides containing that angle by twice the rectangle contained by either of those sides and the projection on it of the other.



Let ABC be a triangle, B an acute angle, BD the projection of AB on BC, then will

$$AC^2 = AB^2 + BC^2 - 2CB \times BD.$$

Proof. For $AC^2 = AD^2 + DC^2$ by Theorem 29,

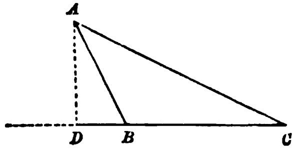
but $AD^2 = AB^2 - BD^2$, by the same Theorem,

and $DC^2 = BC^2 + BD^2 - 2CB \times BD$ (by Theorem 26).

Therefore $AC^2 = AB^2 + BC^2 - 2CB \times BD$.

THEOREM 31.

In an obtuse-angled triangle the square on the side subtending the obtuse angle is greater than the squares on the sides containing that angle by twice the rectangle contained by either of these sides and the projection on it of the other side.



Let ABC be the triangle, ABC being the obtuse angle, BD the projection of AB on BC, BC being produced backward.

Then will $AC^2 = AB^2 + BC^2 + 2CB \, . \, BD$,

for $AC^2 = AD^2 + DC^2$, by Theorem 29,

but $AD^2 = AB^2 - BD^2$,

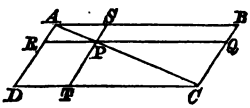
and $DC^2 = CB^2 + BD^2 + 2CB \, . \, BD$, by Th. 25,

therefore $AC^2 = AB^2 + BC^2 + 2CB \, . \, BD$.

EXERCISES ON EQUIVALENT FIGURES.

Ex. 1. *If through any point in the diagonal of a parallelogram lines be drawn parallel to the sides, the two parallelograms so formed through which the diagonal does not pass are equivalent to one another.*

Let $ABCD$ be a parallelogram, P any point on the diagonal AC, and let RPQ, SPT be drawn parallel to the sides.



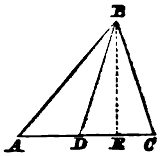
Then will $PB = PD$.

Proof. For the triangle $ABC =$ the triangle ADC (Th. 21) and the triangles ASP, $PQC =$ the triangles ARP, PTC.

Therefore the remainders are equal, that is, $PB = PD$.

These parallelograms are called *complements.*

Ex. 2. *In any triangle the sum of the squares on any two sides is double of the sum of the squares on half the base and on the line which joins the vertex to the middle point of the base.*



Let AC, a side of the triangle ABC, be bisected in D; then will

$$AB^2 + BC^2 = 2AD^2 + 2BD^2.$$

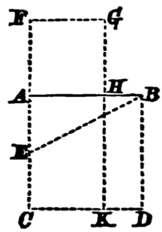
Proof. For let DE be the projection of BD on AC.

Then $AB^2 = AD^2 + DB^2 + 2AD.DE$ (by Theorem 31) and $BC^2 = CD^2 + DB^2 - 2CD.ED$ (by Theorem 30), therefore remembering that $AD = DC$, we obtain by addition that

$$AB^2 + BC^2 = 2AD^2 + 2DB^2.$$

Ex. 3. *To divide a straight line into two parts such that the rectangle contained by the whole line and one of the parts is equal to the square of the other part.*

Let AB be the given line.



Construction. Draw a square $ACDB$ on AB; bisect AC in E. Join BE; produce EA to F, making $EF = EB$; on AF describe a square $AFGH$.

AH and HB are the parts required, so that the rectangle $AB \times BH = AH^2$.

Proof. Produce GH to meet CD in K.

Then since CA is bisected in E, and divided externally in F,

therefore $CF \times FA = EF^2 - EA^2$ (Th. 27, Cor.);

but $EF^2 = EB^2$, and therefore $EF^2 - EA^2 = AB^2$ (Th. 29, Cor. 3),

therefore $CF \times FA = AB^2$;

that is, the figure FK = the figure AD, and therefore

$$FH = HD.$$

But HD is the rectangle $AB \times BH$; and FH is the square on AH;

therefore $AB \times BH = AH^2$.

Exercises.

1. If a straight line be divided into any two parts, the square on the whole line will be equal to the sum of the rectangles contained by the whole line and each of the parts.

2. Construct a square double of a given square.

3. Construct a square equal to two, or three, or any number of given squares.

4. Divide a straight line into two parts, such that the square of one of the parts may be half the square on the whole line.

5. Given the base, area, and one of the angles at the base, construct the triangle.

6. Find the locus of a point which moves so that the sum of the squares of its distance from two given points is constant (Cf. Ex. 2, p. 67).

On the Quadrature of a Rectilineal Area.

There is one more problem which from its historical interest, and from the valuable illustrations it affords of the

methods and limitations of Geometry, should find a place here. This problem is called *the quadrature of a rectilineal area*, which means the finding a square whose area is equivalent to that of any given figure which is bounded by straight lines. It gave a means of comparing any two dissimilarly shaped rectilineal figures, such as irregularly shaped fields whose boundaries were straight. In the present condition of mathematics it is not necessary, as the student will hereafter learn, but it will always be instructive.

The problem is approached by the following stages:

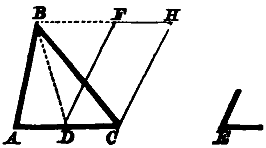
(1) To construct a parallelogram, with sides inclined at a given angle, equivalent to a given triangle.

(2) To construct *on a given straight line* a parallelogram, with sides inclined at a given angle, equivalent to a given triangle.

(3) To construct a parallelogram, with sides inclined at a given angle, equivalent to a *given rectilineal figure*.

(4) To construct a *square* equivalent to a given rectilineal figure.

(1) *To construct a parallelogram, with sides inclined at an angle = E, equal to the triangle ABC.*



Construction. Bisect AC in D, make the angle $CDF = E$, and through B draw BFH parallel to AC, and draw CH parallel to DF.

$FDCH$ will be the parallelogram required.

Proof. If BD be joined, it will be clear that the triangle BAC and the parallelogram $FHCD$ are each of them double of the triangle BDC (Th. 23), and therefore the

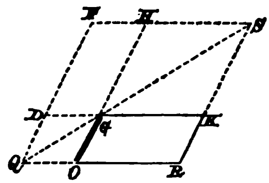
parallelogram $FHCD$ = the triangle BAC and it has an angle = E, which was required.

It is now possible to proceed to the second stage of the problem, viz.

(2) *To construct on a given straight line, a parallelogram, having a given angle, equal to a given triangle.*

Let BAC be the given triangle, E the given angle as before, and let it be required to construct on the line GO a parallelogram equivalent to BAC, and having an angle E.

Construction. Construct the parallelogram $FDGH$ as before, and place it so that one of its sides GH may be in the same straight line with GO.



Produce FD, and draw OQ parallel to GD to meet FD in Q. Join QG, and produce it to meet FH produced in S.

Draw SKR parallel to FQ, meeting DG produced in K, and QO produced in R.

Then $GORK$ is the parallelogram required.

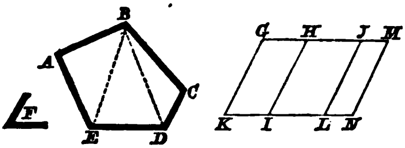
Proof. For the parallelogram FG = the parallelogram GR, being complements; and FG = the given triangle ABC.

Therefore GR = the triangle ABC, and it has an angle = E, since it is equiangular with the parallelogram $FDGH$.

The third stage of the problem consists in a repeated application of these two constructions.

(3) *To construct a parallelogram, having a given angle, equivalent to a given rectilineal figure.*

Let $ABCDE$ be the given rectilineal figure; F, the given angle. Divide $ABCDE$ into triangles by joining BE, BD.



Construction. Construct as before a parallelogram $GHIK = BAE$, and having an angle at $K = F$.

Construct on HI a parallelogram, $HJLI = BED$, and having the angle $HIL = K$.

And construct on JL a parallelogram $JMNL = BCD$ and having the angle $JLN = F$.

$GKNM$ will then be the parallelogram required.

Proof. For since the angle $HIL =$ the angle K, it is therefore supplementary to HIK; and therefore (by Theorem 1) KIL is a straight line.

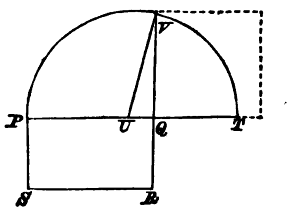
Similarly GM and KN are straight lines; and MN is obviously parallel to GK.

Therefore $GKNM$ is a parallelogram, having the given angle, and it is by construction equivalent to the given rectilineal figure.

Now we are able to solve the original problem.

(4) *Let it be required to construct* a square $= ABCDE$.

Construction. By the previous construction make a *rectangle* equal to $ABCDE$, and let $PQRS$ be the rectangle so made.



Then if $PQ = QR$ the rectangle is a square; but if not, produce PQ to T, making $QT = QR$; on PQT as

diameter describe a semicircle, U being the centre, and produce RQ to meet the circumference in V.

If a square be described on QV, this square will be equal to $ABCDE$.

Proof. For since PQ is the sum of PU and UQ, and QT is the difference of PU (or UT) and UQ, it follows (from Theorem 27) that the rectangle $PQ \times QT = PU^2 - UQ^2$; but $PU^2 = UV^2$, and therefore $PU^2 - UQ^2 = UV^2 - UQ^2$, that is, VQ^2, by Theorem 30.

But the rectangle $PQ \times QT$ is the rectangle $PQRS$, which was made equal to $ABCDE$.

Therefore $VQ^2 = ABCDE$, and the square described on VQ is the square required.

Remark. If the given figure is not rectilineal, it cannot be divided into triangles: hence by this method it is impossible to construct a square equal to a given curvilinear area. Nor can any method depending on the use of the ruler and compasses only, (see p. 38,) construct a square equal to some curvilinear areas, such as the circle. This is the problem of squaring the circle, the solution of which cannot be effected without the use of other instruments.

We subjoin a few problems and theorems as miscellaneous exercises in the Geometry of angles, lines, triangles, parallelograms, and the equivalence of figures.

Miscellaneous Theorems and Problems.

1. Prove that the acute angle between the bisectors of the angles at the base of an isosceles triangle is equal to one of the angles at the base of the triangle.

2. Find a point equally distant from three given straight lines.

3. Find the locus of the middle point of a line drawn from a given point to meet a given line.

4. If the diagonals of a quadrilateral bisect one another and are equal to one another, the figure will be a rectangle.

5. If the diagonals of a quadrilateral bisect one another at right angles and are also equal, the figure will be a square.

6. If ABC is a triangle, AB being greater than AC, and a point D in AB be taken such that $AD = AC$; prove that BCD is equal to half the difference of the angles ABC, ACB.

7. If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.

8. If $ABCD$ is a parallelogram, and $AE = CF$ are cut off from the diagonal AC, then $BEDF$ will be a parallelogram.

9. If $AA' = CC'$ be cut off from the diagonal AC, and $BB' = DD'$ from the diagonal BD of a parallelogram, then will $A'B'C'D'$ be also a parallelogram.

10. If $AA' = BB' = CC' = DD'$ be cut off from the sides of the parallelogram $ABCD$ taken in order, then will $A'B'C'D'$ be also a parallelogram.

11. ABC is a triangle, and through D, the middle point of AB, DE, DF are drawn parallel to the sides BC, AC to meet them in EF. Shew that EF is parallel to AB.

12. Through a given point to draw a line such that the part of it intercepted between two parallel lines shall have a given length.

13. To describe a rhombus equal to a given parallelogram, having its side equal to the longer side of the parallelogram.

14. Shew that the diagonal of a rectangle is longer than any other line whose extremities are on the sides of the rectangle.

15. From the extremities of the base of an isosceles triangle straight lines are drawn perpendicular to the opposite sides; the angles made by them with the base are equal to half the vertical angle.

16. If one angle of a triangle is equal to the sum of the other two, the greatest side is double of the distance of its middle point from the opposite angle.

17. Find the locus of a point, given the sum or difference of its distances from two fixed lines.

18. ABC is a triangle, AB greater than BC; BD bisects the base AC, and BE the angle ABC. Prove (1) that ADB is an obtuse angle; (2) that ABD is less than DBC; and (3) that BE is less than BD.

19. If two sides of a triangle be given, its area will be greatest when they contain a right angle.

20. BCD... are points on the circumference of a circle, A any point not the centre of the circle. Shew that of the lines AB, AC, AD... not more than two can be equal.

21. Of all triangles having the same base and area, that which is isosceles has the least perimeter.

22. Of all triangles having the same vertical angle, and whose bases pass through a given point, the least is that whose base is bisected in that point.

23. The diagonals of a parallelogram divide it into four equivalent triangles.

24. If from any point in the diagonal of a parallelogram straight lines be drawn to the angles, then the parallelogram will be divided into two pairs of equivalent triangles.

25. $ABCD$ is a parallelogram, and E any point in the diagonal AC produced. Shew that the triangles EBC, EDC will be equivalent.

26. $ABCD$ is a parallelogram, and O any point within it, shew that the triangles OAB, OCD are together equivalent to half the parallelogram.

27. On the same supposition if lines are drawn through O parallel to the sides of the parallelogram, then the difference of the parallelograms DO, BO is double of the triangle OAC.

28. The diagonals of a parallelogram intersect in O, and P is a point within the triangle OAB. Prove that the difference of the triangles APB, CPD, is equivalent to the sum of the triangles APC, BPD.

29. If the points of bisection of the sides of a triangle be joined, the triangle so formed shall be one-fourth of the given triangle.

30. Shew that the sum of the squares on the lines

joining the angular points of a square to any point within it is double of the sum of the squares on the perpendiculars from that point on the sides.

31. If the sides of a quadrilateral figure be bisected, and the points of bisection joined, prove that the figure so formed will be a parallelogram equal in area to half the given quadrilateral.

32. Any line drawn through the intersection of the diagonals of a parallelogram to meet the sides bisects the figure.

33. D is the middle point of the side AC of a triangle ACB, and any parallel lines BE, DF are drawn to meet AC, AB (or BC) in E and F, shew that EF divides the ria ngle into two equal areas.

34. If the sides of a triangle are 3, 4, 5 inches respectively, the triangle is right-angled.

35. The area of a rhombus is equal to half the rectangle constructed on the two diameters of the rhombus.

36. If two opposite sides of a quadrilateral are parallel, and their points of bisection joined, the quadrilateral will be bisected.

37. If two opposite sides of a parallelogram be bisected, and lines be drawn from these two points of bisection to the opposite angles, these lines will be parallel, and will trisect the diagonal.

38. The sum of the squares described on the sides of a rhombus is equal to the squares described on its diameters.

39. From the sides of the triangle ABC, AA', BB', CC', are cut off each equal to two-thirds of the side from which it is cut. Shew that the triangle $A'B'C'$ is one-third of the triangle ABC.

40. Find the locus of the vertices of triangles of equal area upon the same base.

41. Find the locus of a point, such that the sum of the squares on its distances from two given points is equal to the square on the distance between the two points.

42. If m and n are any numbers, and lines be taken whose lengths are m^2+n^2, m^2-n^2 and $2mn$ units respectively, shew that these lines will form a right-angled triangle. Give examples of these triangles.

43. Through two given points on opposite sides of a straight line draw two straight lines to meet in that line, so that the angle which they form shall be bisected by that line.

44. Through a given point draw a line such that the perpendiculars on it from two given points may be equal.

45. Find points D, E in the equal sides AB, AC of an isosceles triangle ABC, such that $BD=DE=EC$.

46. Given two points and a straight line of indefinite length, construct an equilateral triangle so that two of its sides shall pass through the given points, and the third shall be in the given straight line.

47. Construct an isosceles triangle having the angle at the vertex double of the angles at the base.

48. Bisect a triangle by a line passing through one of its angular points.

49. Bisect a triangle by a line passing through a point in one of its sides.

50. Bisect a parallelogram by a line passing through any given point.

51. Construct a triangle equal to a given quadrilateral figure.

52. Bisect a given quadrilateral figure by a line drawn from one of its angular points.

53. Bisect a given five-sided figure by a line drawn from one of its angular points.

54. Produce a given straight line to such a distance that the square on the produced part may be double of the square on the given line.

55. Produce a given straight line to such a distance that the square on the whole line may be double of the square on the given line.

56. Given two sides and a median, construct the triangle.

57. Divide a straight line into two parts such that the square on one part may be four times the square on the other.

58. From B, one of the angles of a triangle ABC, a perpendicular BD is let fall on AC. Shew that the difference of the squares on AB, BC is equal to the difference of the squares on AD, DC.

59. AC one of the sides of a triangle ABC is bisected in D: and BD joined. Shew that the squares on AB and BC together are equal to twice the square on BD, and twice the square on AD.

60. Produce a given line AB to P so that $AP \cdot BP = AB^2$.

61. $ABCD$ is the diameter of two concentric circles, P, Q any points on the outer and inner circles respectively. Prove that $BP^2 + CP^2 = AQ^2 + DQ^2$.

62. Prove that the squares on the diagonals of a rectangle are together equal to the squares on its sides.

63. Prove that the squares on the diagonals of any parallelogram are together equal to the squares on its sides.

64. O is the point of intersection of the diagonals of a square $ABCD$, and P any other point whatever. Prove that $$AP^2 + BP^2 + CP^2 + DP^2 = 4OA^2 + 4OP^2.$$

65. Given the base, difference of sides, and difference of angles at the base, construct the triangle.

66. If from one of the acute angles of a right-angled triangle a line be drawn to the opposite side, the squares on that side and the line so drawn are together equal to the squares on the segment adjacent to the right angle and on the hypothenuse.

67. If from the right angle C of a right-angled triangle ABC straight lines be drawn to the opposite angles of the square on AB, the difference of the squares on these two lines will equal the difference of the squares on AC and BC.

68. AB is divided into two unequal parts in C and equal parts in D; shew that the squares on AC and BC are greater than twice the rectangle $AC \times CB$ by four times the square on CD.

69. In any right-angled triangle the square on one of the sides containing the right angle is equal to the rectangle contained by the sum and difference of the other two sides.

70. In any isosceles triangle ABC, if AD is drawn from A the vertex to any point D in the base, shew that
$$AB^2 = AD^2 + BD \, . \, DC.$$

71. Prove that four times the sum of the squares on the medians of a triangle is equal to three times the sum of the squares on the sides of the triangle.

72. The square of the base of an isosceles triangle is double the rectangle contained by either side, and the projection on it of the base.

73. The squares on the diagonals of a quadrilateral are double of the squares on the sides of the parallelogram formed by joining the middle points of its sides.

74. Hence shew that they are also double of the squares on the lines which join the points of bisection of the opposite sides of the quadrilateral.

75. The squares on the diagonals of a quadrilateral are together less than the squares on the four sides by four times the square on the line joining the points of bisection of the diagonals.

76. In any quadrilateral figure the lines which join the middle points of opposite sides intersect in the line which joins the middle point of the diagonals.

77. The locus of a point which moves so that the sum of the squares of its distances from three given points is constant is a circle.

BOOK II. THE CIRCLE.

INTRODUCTION.

Def. 1. IF a point moves in a plane so that its distance from a fixed point is constant, it traces out a line which is called *the circumference of a circle.*

Def. 2. The fixed point is called *the centre* of the circle.

Def. 3. The distance of any point on the circumference from the centre is called *the radius.*

Def. 4. A line through the centre, terminated both ways by the circumference, is called *a diameter* of a circle.

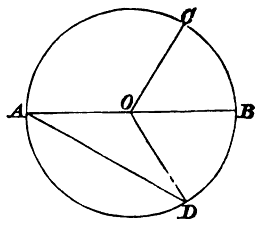
Def. 5. Any portion of a circumference is called *an arc.*

Def. 6. The figure enclosed by an arc and the radii to its extremities is called *a sector.*

Def. 7. The line joining the extremities of an arc is called *a chord* of that arc.

Def. 8. The parts into which a chord divides a circle are called *segments.*

Several properties of a circle follow at once from the definitions.



1. *All radii of a circle are equal.*

2. *All diameters of a circle are equal.*

3. *A circle cannot cut a straight line in more points than two: for from the centre of the circle not more than two equal straight lines can be drawn to meet the given straight line.*

4. *If a circle were to rotate round its centre its circumference would always occupy the same position.*

5. *Any arc is superposable on an equal arc of the same circle.*

6. *Circles are equal whose radii are equal.*

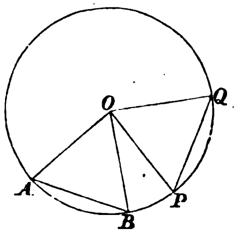
7. *A point is outside, on, or inside the circumference according as its distance from the centre is greater than, equal to, or less than the radius.*

SECTION I.

PROPERTIES OF CENTRE.

THEOREM I. ROTATORY PROPERTIES OF THE CIRCLE.

Equal arcs of a circle subtend equal angles at the centre, and have equal chords; and conversely, equal angles at the centre cut off equal arcs and have equal chords; and equal chords in a circle cut off equal arcs, and subtend equal angles at the centre.



Let $ABPQ$ be a circle, and let the *arc* AB be given equal to arc PQ, then will the angle AOB at the centre = the angle POQ, and the chord AB = the chord PQ.

For if the sector AOB were to rotate round O till A fell on P, B would fall on Q, (since arc AB = arc PQ), and the angle AOB would coincide with POQ, and therefore is equal to it; and for the same reason the chord AB = the chord PQ.

In the same manner it may be shewn that if the *angle* AOB is given equal to POQ, then the arcs and chords would coincide and are therefore equal.

And if the *chord* AB is given equal to the chord PQ, then the triangles AOB, POQ have the three sides of the one equal to the three sides of the other; and therefore the

angle AOB = the angle POQ; and therefore also the arc AB = the arc PQ.

Cor. 1. *If the arcs or angles of sectors of a circle are equal, the sectors are equal.*

Cor. 2. *In equal circles equal arcs subtend equal angles at the centre and cut off equal chords.*

Cor. 3. *The diameter divides the circle into two equal parts, which are therefore called semicircles.*

Remark on Theorem 1.

The student has now had several examples of theorems, and their *converse* and *opposite* theorems, and it will be well for him to observe *under what conditions a converse theorem is true.*

This may be generalized into the following statement. If A, B, C... as conditions involve D as a result, and the failure of C involves a failure of D; then A, B, D... as conditions involve C as a result.

For example, in a circle, (A), angles at the centre, (B) which stand on equal arcs, (C), are equal, (D),

and if the arcs are unequal, {the failure of C}, the angles are unequal, {the failure of D}.

Hence it follows logically that in a circle, (A), angles at the centre, (B), which are equal (D), stand on equal arcs, (C).

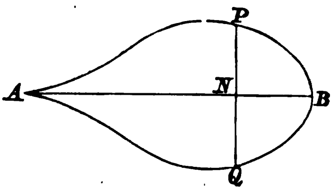
All converse theorems can be at once proved by the reductio ad absurdum, but we shall in general enunciate immediate converses, when they are true, without giving a detailed proof.

Symmetry of a figure with respect to a line.

Def. 9. A figure is said to be *symmetrical with respect to a line*, when every line at right angles to that line cuts the figure at points which are equidistant from that line.

Thus the figure $APBQ$ is symmetrical with respect to AB, if every line PNQ at right angles to AB cuts it so that

$$PN = QN.$$

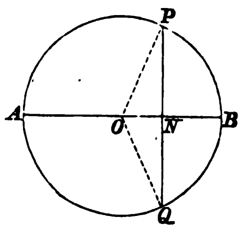


THEOREM 2.

SYMMETRY OF THE CIRCLE WITH RESPECT TO ITS DIAMETER.

A circle is symmetrical with respect to any diameter.

Let AB be any diameter, PNQ any line drawn perpendicular to AB, meeting the circle in P and Q. Then shall $PN = NQ$.

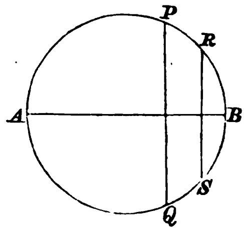


For OP and OQ are equal obliques drawn from O to PQ, and therefore they are equally distant from the perpendicular: and therefore $PN = NQ$ (I. 12).

COR. 1. *Hence the semicircle APB if folded over the line AB, would coincide with the semicircle AQB, PN falling on NQ; and therefore the arc AP = the arc AQ, and the arc PB = the arc BQ.*

COR. 2. *Hence also parallel chords in a circle intercept equal arcs.*

If PQ, RS are parallel chords, then will the arc PR = the arc QS.



For when one semicircle is folded on the other, P would coincide with Q, and R with S, and therefore the arc PR with the arc QS.

COR. 3. This fundamental property of a circle, viz. its symmetry with respect to a diameter, gives rise to many theorems. For it appears that the diameter AB in the figure of Cor. 2 fulfils six conditions: (1) It is perpendicular to the chord PQ; (2) it passes through the centre; (3) it bisects the chord; (4) it bisects the arc QAP; (5) it bisects the arc PBQ; (6) it bisects any chord parallel to PQ. *And since only one line can be drawn to fulfil any two of the above conditions, it follows that a line which fulfils any two of them, fulfils the remaining four.*

For example: the original theorem, with its first corollary, is (α) *If a line fulfils* (1) *and* (2), *it also fulfils* (3), (4) *and* (5).

Hence (β) *If a line bisects a chord at right angles* (1) *and* (3), *it must pass through the centre* (2).

And, (γ) *A line that bisects any chord and its arc,* (3) *and* (4), *will pass through the centre* (2).

And, (δ) *The line drawn from the centre to bisect a chord* (2 *and* 3) *is perpendicular to that chord* (1).

By combining these data in different ways, many different theorems may be made.

Exercises.

1. If a straight line cut two concentric circles, the parts of it intercepted between the two circumferences will be equal.

2. Of two angles at the centre, the greater angle is subtended by the greater arc; and also by the greater chord, if the sum of the two angles is less than four right angles. Prove this and its converse propositions.

3. Perpendiculars are let fall from the extremities of a diameter on any chord, or any chord produced; shew that the feet of the perpendiculars are equally distant from the centre.

4. The locus of the points of bisection of parallel chords of a circle is the diameter at right angles to those chords.

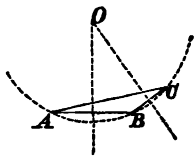
5. If a diameter of a circle bisects a chord which does not pass through the centre, it will bisect all chords which are parallel to it.

6. AB and CD are unequal parallel chords in a circle, prove that AC and BD, and likewise AD and BC intersect on the diameter perpendicular to AB and CD, or that diameter produced, and are equally inclined to that diameter.

What will be the case if AB and CD are equal?

THEOREM 3.

One circle, and only one circle, can be drawn to pass through three given points which are not in the same straight line.



Let A, B, C be the three given points. Join AB, BC.

Then since AB is to be a chord, the locus of the centre is the straight line that bisects AB at right angles (II. 2, β).

Similarly, the line that bisects BC at right angles must pass through the centre. Hence the centre must be at O, the point of intersection of these perpendiculars; and the circle described with centre O and radius OA will pass through A, B and C.

And there can be only one centre, since the perpendiculars intersect in only one point.

The three points thus *determine* the circle.

COR. 1. *Circles that have three points in common, coincide wholly.*

Hence a circle is named by the letters which mark three points on its circumference.

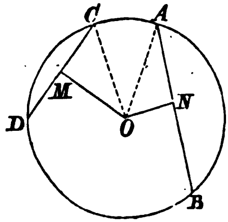
COR. 2. *Different circles can intersect in two points only.*

Def. 10. The circle is said to be *circumscribed* about the triangle ABC, and the triangle ABC is said to be *inscribed* in the circle, when the points A, B, C are on the circumference of the circle.

Def. 11. *The distance* of a chord from the centre is the perpendicular on the chord from the centre.

THEOREM 4.

Equal chords of a circle are equally distant from the centre, and conversely; and of two unequal chords the greater is nearer to the centre than the less, and conversely.



Let AB, CD be chords of a circle, OM, ON the perpendiculars on them from the centre, bisecting the chords in M and N respectively. Join OC, OA.

Then since M and N are right angles, therefore

$$OM^2 + MC^2 = OC^2 \quad (\text{I. } 29),$$

and $$ON^2 + NA^2 = OA^2,$$

but $$OC = OA, \text{ and } OC^2 = OA^2;$$

therefore $$OM^2 + MC^2 = ON^2 + NA^2.$$

Hence, (1) if $AB = CD$ and $AN = CM$,

it follows that $$OM = ON.$$

(2) If $AB > CD$, $AN > CM$,

and therefore OM is $< ON$.

(3) If $ON = OM$,

therefore $AN = CM$ and $AB = CD$.

(4) If $ON < OM$,

$AN > CM$ and $AB > CD$.

COR. 1. *The diameter is the greatest chord of a circle.*

COR. 2. *The locus of the middle points of equal chords in a circle is a concentric circle.*

Exercises.

1. Given a triangle ABC to find the centre of the circumscribing circle.

2. A chord 8 inches long is drawn in a circle whose radius is 5 inches; find the distance of the chord from the centre.

3. A chord is drawn at the distance of one foot from the centre of a circle whose diameter is 26 inches; find the length of the chord.

4. Given a circle to find its centre.

5. If two equal chords intersect one another, the segments of the one are equal to the segments of the other respectively.

6. Two chords cannot bisect one another unless both pass through the centre.

7. Given a curve, to ascertain whether it is an arc of a circle or not.

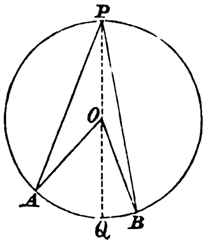
SECTION II.

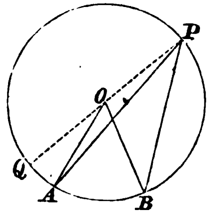
ANGLES IN SEGMENTS OF THE CIRCLE.

THEOREM 5.

The angle subtended at any point in the circumference by any arc of a circle is half of the angle subtended by the same arc at the centre.

Let AB be any arc, O the centre, P any point on the





circumference. Then will the angle AOB be double of the angle APB.

Join PO, and produce it to Q.

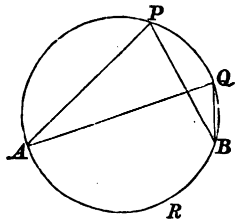
Then because, from the definition of a circle, OPA is an isosceles triangle, the angle OAP = the angle OPA: but the exterior angle AOQ is equal to the two interior and opposite angles OAP and OPA (I. 7); and therefore the angle AOQ is double of the angle OPA. Similarly the angle QOB is double of the angle OPB.

Hence (in fig. 1) the sum, or (in fig. 2) the difference of the angles AOQ, QOB is double of the sum or dif-

ference of OPA and OPB, that is, the angle AOB is double of the angle APB, or the angle APB is half of the angle AOB.

Def. 12. The angle APB is said to be the angle *in the segment* APB.

Cor. 1. *Angles in the same segment of a circle are equal to one another.*

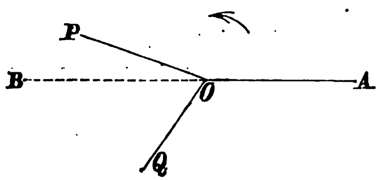


For each of the angles APB, AQB is half of the angle subtended at the centre by the arc ARB.

The segment $APQB$ is said to be *capable* of an angle equal to the angle APB or AQB.

Remark. It is important to remember here that angles may be greater than two right angles, and that the existence of such angles is contemplated in the foregoing theorem and corollary.

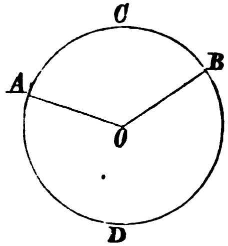
Thus, if a line be conceived to revolve round O starting from an initial position OA, and revolving in the



direction of the arrow, it describes an angle of constantly increasing magnitude; when it reaches the position OB it has described an angle equal to two right angles; and in the position OQ it has described an angle greater than

two right angles. There are thus two angles AOQ, one taken in the direction of the arrow greater than two right angles, and the other taken in the opposite direction less than two right angles.

Thus, in the figure, the angle BOA standing on the arc ACB is less than two right angles; and the angle AOB on the arc ADB is greater than two right angles.

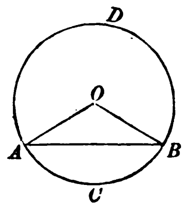


Hence it is clear that the angle at the centre standing on any arc is less than, equal to, or greater than two right angles according as the arc is less than, equal to, or greater than a semi-circumference.

COR. 2. *If a circle is divided into any two segments by a chord, the angles in the segments will be supplementary to one another.*

For of the two angles at O, the one is double of the angle in the segment ADB, and the other of the angle in the segment ACB.

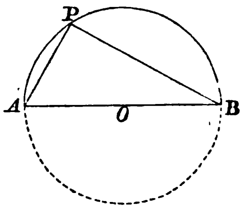
But the angles at O make up four right angles (I. 2), therefore the angles in the segments ACB, ADB make up two right angles.



These segments are called *Supplementary Segments.*

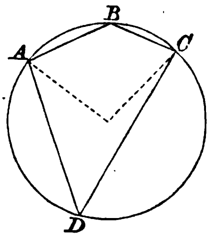
COR. 3. *The angle in a semicircle is a right angle.*

For let APB be a semicircle: then the angle in the segment is half the angle at the centre: that is, the angle APB is half the angle AOB, which is two right angles; therefore the angle APB is a right angle.



Def. 13. A polygon is said to be *inscribed in* a circle, when its angular points are on the circumference of the circle.

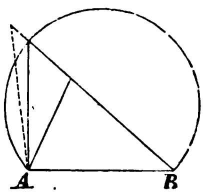
Cor. 4. *The opposite angles of every quadrilateral figure inscribed in a circle are together equal to two right angles.*



For the segments ABC, ADC are supplementary segments, and so also are the segments BAD, BCD.

Therefore by Cor. 3 the angles ABC, ADC are together equal to two right angles; and the angles BAD, BCD are also together equal to two right angles.

Cor. 5. *The locus of a point at which a given straight line subtends a constant angle is an arc of a circle.*



For if AB be the given straight line, and on it there is described a segment capable of the given angle, at every point in the arc the straight line subtends the given angle, and at every other point the angle is greater or less, according as it is within or without the segment.

The segment may be described on both sides of the line AB.

It will be seen that this is the converse and opposite of Cor. 1.

Cor. 6. *In equal circles equal angles at the circumferences stand upon equal arcs, and conversely.*

This may be proved by superposition.

Exercises.

1. Prove that the lines which join the extremities of equal arcs in a circle are either equal or parallel.

2. If two opposite angles of a quadrilateral figure are together equal to two right angles, prove that a circle which passes through three of its angular points will also pass through the fourth.

3. If two opposite sides of a quadrilateral inscribed in a circle are equal, prove that the other two are parallel.

4. AB, CD are chords of a circle which cut at a constant angle. Prove that the sum of the arcs AC, BD remains constant, whatever may be the position of the chords.

5. If the diameter of a circle be one of the equal sides of an isosceles triangle, prove that its circumference will bisect the base of the triangle.

6. Circles are described on two sides of a triangle as diameters. Prove that they will intersect on the third side or third side produced.

7. Any number of chords of a circle are drawn through a point on its circumference: find the locus of their middle points.

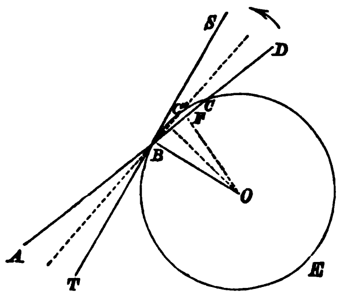
8. If through any point, within or without a circle, lines are drawn to cut the circle, prove that the locus of the middle points of the chords so formed is a circle.

SECTION III.

THE TANGENT AND NORMAL.

Def. 14. When a straight line cuts a circle it is called a *secant.*

Thus $ABCD$ is a secant of the circle ACE.



Def. 15. When one of the points in which a secant cuts a circle is made to move up to, and ultimately coincide with the other, the ultimate position of the secant is called *the tangent* at that point.

Thus, if $ABCD$ be conceived to revolve round B, in the direction of the arrow, the point C will move to C' and will ultimately coincide with B, and the line TBS, which is the position the secant then attains, is said to be a tangent to the circle at the point B.

The point B is then called *the point of contact.*

Several important properties follow at once from this definition of a tangent.

Theorem 6.

A tangent meets the circle in one point only, viz. the point of contact.

For since a secant can cut a circle in two points only, it follows that the parts AB, CD are wholly without the circle; and therefore when C moves up to B, and the chord BC is merged in the point B, the whole line, with exception of the point B, is outside the circle.

Theorem 7.

The radius to the point of contact is at right angles to the tangent.

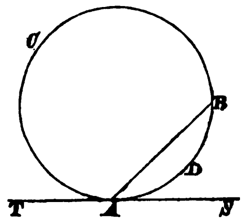
For if F be the middle point of the chord BC, OF is perpendicular to BC; and as C moves to B, F will also move up to B, and when the secant becomes a tangent, OF, which is always at right angles to the secant, coincides with the radius OB.

Therefore OB is at right angles to the tangent TBS.

Cor. 1. *Hence there can be only one tangent to a circle at a given point.*

Cor. 2. *The line at right angles to the tangent through the point of contact passes through the centre.*

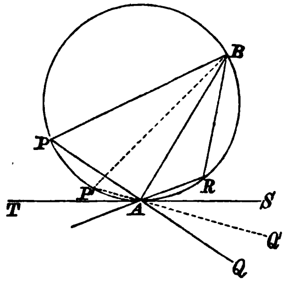
Def. 16. When a secant is drawn from the point of contact of a tangent it divides the circle into segments which are said to be *alternate* to the angles made by the tangent with the secant on its sides opposite to the segments.



Thus ACB is a segment alternate with BAS, and BDA with BAT.

Theorem 8.

If from the point of contact of a straight line and a circle a chord of the circle be drawn, the angles made by the chord with the tangent will be equal to the angles in the alternate segments of the circle.



Let a chord AB be drawn from the point of contact A of the tangent TAS.

Then will the angles BAS, BAT be equal to the angles in the alternate segments of the circle.

For take any point P in the arc of the segment alternate to BAS, and join PB, PA, and produce PA to Q.

Conceive the point P to move along the arc towards A. Then the angle BPA in the segment remains always the same; and it will after a while assume the position of the dotted lines $BP'A$, and ultimately when P has moved up to A, the angle BPA will coincide with the angle BAS, since the limiting position of the secant PQ is the tangent AS, and BP then coincides with BA. Therefore the angle BAS = the angle BPA in the alternate segment.

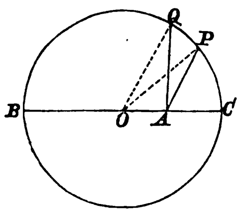
Similarly it may be shewn that the angle BAT = the angle BRA.

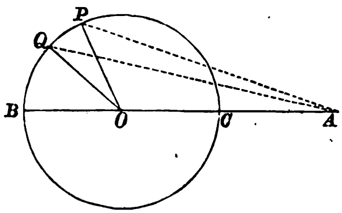
Def. 17. If from any point on a curve a line is drawn at right angles to the tangent at that point, it is called a *normal to the curve at that point.*

Since the radius is at right angles to the tangent to a circle, it follows that all radii are normals to a circle.

THEOREM 9.

From any point within or without a circle except the centre, two and only two normals can be drawn, one of which is the shortest, and the other the longest line that can be drawn from that point to the circumference: and as a point moves along the circumference from the extremity of the shortest to the extremity of the longest normal, its distance from the fixed point continually increases.





Let A be the fixed point, O the centre, and let AO produced through the centre meet the circumference in B, and produced if necessary in the other direction meet it in C.

Then AB and AC are normals and are the only normals, and are respectively the longest and shortest lines that can be drawn from A to the circumference: and if P, Q are any other points such that the arc CP is less than CQ, AP shall be less than AQ.

Join OP, OQ.

Then it is clear that AB and AC are at right angles to the tangents at B and C, since A is a point in the radius OB or OC.

And if P is any other point, OP is the normal at that

point, and therefore AP is not the normal: hence two and only two normals can be drawn.

Again, in the triangle APO, the difference of OP and OA is AC, since $OP = OC$; and therefore AC is less than AP (I. 13): and the sum of OP and OA is AB, since $OP = OB$, and therefore AB is greater than AP. Hence of all lines drawn from A to the circumference AC is the least and AB the greatest.

Lastly, in the triangles AOP, AOQ since the two sides AO, OP are equal to AO, OQ, but the angle AOQ greater than the angle AOP, therefore AQ is greater than AP (I. 17).

Therefore as a point moves from C to A along the arc, its distance from A continually increases.

Cor. 1. *Two and only two equal straight lines can be drawn from A to the circumference, one on each side of the shortest normal.*

Cor. 2. *A point from which more than two equal straight lines can be drawn to a circumference must be the centre.*

Theorem 10.

Intersection of Circles.

The line that joins the centres of two intersecting circles, or that line produced, bisects at right angles their common chord.

Fig. 1.

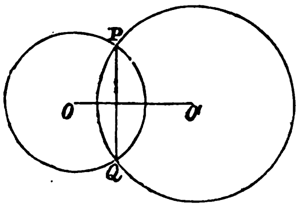
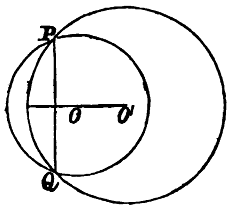


Fig. 2.



Let O, O' be the centres of two intersecting circles, PQ their common chord, then shall OO' or OO' produced bisect PQ at right angles.

For since PQ is a chord of both circles, the line which bisects PQ at right angles passes through both centres (II. 2, β); that is, it must be the line OO'.

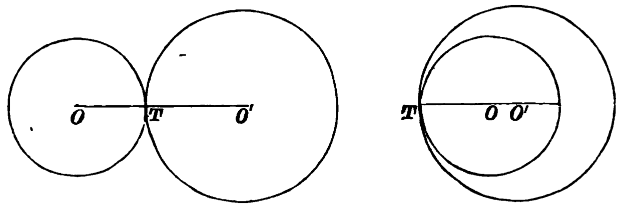
CONTACT OF CIRCLES.

Def. 18. When one of the points in which one circle cuts another moves up to and ultimately coincides with the other, *the circles are said to touch one another* at that point.

Since two circles intersect in only two points, it follows that two circles which touch one another can have no other point in common; for the two points of intersection are merged in the point of contact.

THEOREM 11.

If two circles touch one another, the line that joins their centres will pass through the point of contact.



For if in the figures of Th. 10, the centres O, O' of the circles were to recede from one another, or were to approach one another, the points P and Q would after a while approach one another, and the chord PQ would become indefinitely small, and be merged in the point T, and the circles would touch one another at T by the definition.

But the line OO' always bisects PQ, and therefore it will ultimately pass through T the point of contact.

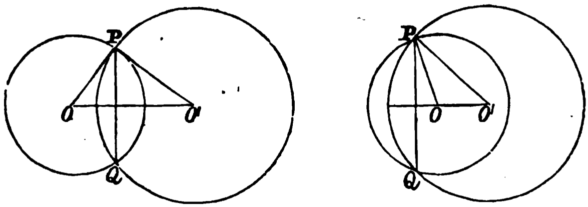
COR. I. *Two circles that touch one another have a common tangent at their point of contact.*

For the line at right angles to OO' through T is a tangent to both circles by Th. 7.

COR. 2. *If R, r are the radii of two circles, D the distance between their centres, it follows that*

(1) *When the circles intersect,*

$$R+r>D \text{ or } R-r<D. \quad (\text{I. } 13).$$



(2) *When the circles touch,*

$$R+r=D \text{ or } R-r=D.$$

(3) *When the circles do not meet,*

$$R+r<D \text{ or } R-r>D.$$

In other words, if R, r, D are such that any two of them are greater than the third, the circles will intersect; if two of them are together equal to the third, the circles will touch; and if two of them are together less than the third, the circles will not meet, but be wholly inside or wholly outside one another.

EXERCISES.

1. If a straight line touch the inner of two concentric circles, and be terminated by the outer, prove that it will be bisected at the point of contact.

2. Any two chords which intersect on a diameter and make equal angles with it are equal.

3. Two circles touch each other externally, and a third circle is described touching both externally. Shew that the difference of the distances of its centre from the centre of the two given circles will be constant.

4. If two circles intersect one another, and circles are drawn to touch both, prove that either the sum or the difference of the distances of their centres from the centres of the fixed circles will be constant, according as they touch (1) one internally and one externally, (2) both internally or both externally.

5. If two circles touch one another, any line through the point of contact will cut off segments from the two circles capable of the same angle.

6. If two circles touch one another, two straight lines through the point of contact will cut off arcs, the chords of which are parallel.

7. Two circles cut one another, and lines are drawn through the points of section and terminated by the circumference, shew that they intercept arcs the chords of which are parallel.

8. Circles whose radii are 6·7 and 7·8 inches are successively placed so as to have their centres 14, 14½, and 15 inches apart. Shew whether the circles will meet or touch or not meet one another.

9. What will be the case if the centres are 1 inch, 1·1 inch, or 1·2 inches apart?

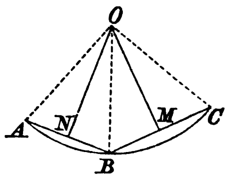
SECTION IV.

PROBLEMS.

PROBLEM 1.

Given an arc of a circle, to find the centre of the circle of which it is an arc.

Let ABC be the arc.



Construction. Draw any two chords AB, BC, and bisect them at right angles by straight lines ON, OM, intersecting at O (I. 3, 2). O shall be the centre required.

Proof. For NO is the locus of points equidistant from A and B, and therefore $AO = BO$.

Similarly, MO is the locus of points equidistant from B and C; therefore O is equidistant from A, B and C.

Hence, the circle described with centre O and radius equal to one of these three lines, will pass through the other two, and having three points coinciding with the given circular arc, must coincide with it throughout.

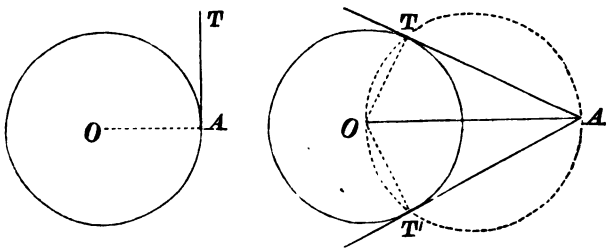
PROBLEM 2.

To draw a tangent to a circle from a given point.

There will be two cases.

First, let the given point A be on the circumference. Let O be the centre.

Construction. Join OA, and draw AT at right angles to OA (I. 2).



Proof. Then AT is a tangent by Th. 7.

Secondly, let A be outside the circle.

Construction. On OA as diameter describe a circle, cutting the given circle in T and T'. Join AT, AT'; these shall be tangents from A.

Proof. For join OT, OT'. Then since ATO is a semicircle, the angle ATO is a right angle (II. 5, 2). That is, AT or AT' is at right angles to the radius to the point where it meets the circumference, and therefore AT and AT' are tangents.

It may easily be proved that $AT = AT'$.

PROBLEM 3.

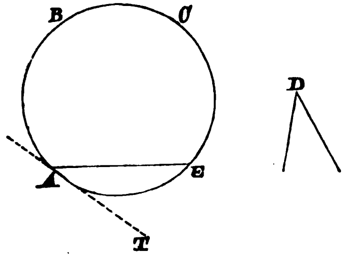
To cut from any circle a segment which shall be capable of a given angle.

Let ABC be the circle, D the given angle.

Construction. Take any point A on the circumference.

Draw AT the tangent at A (II. 2); and make an angle TAE at A equal to the angle D (I. 5, Cor. 2).

Then shall AE be the chord of the segment required.

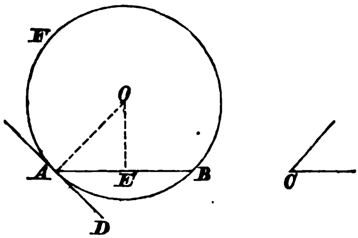


Proof. For the angle in the segment alternate to TAE is equal to the angle TAE, that is, is equal to D.

PROBLEM 4.

On a given straight line to describe a segment of a circle containing an angle equal to a given angle.

Let AB be the given line, C the given angle.



Construction. At the point A make an angle BAD equal to the angle C (I. 5, Cor. 2).

Then if a circle be described to touch AD in A, and to pass through B, the segment of that circle alternate to BAD will be the segment required.

To find the centre of this circle, draw AO at right angles to AD: then AO is the locus of the centres of all circles which touch AD at A.

And bisect AB at right angles by the line EO; then EO is the locus of the centres of circles which pass through A and B.

Therefore O, the point of intersection of these lines, is the centre of the circle required.

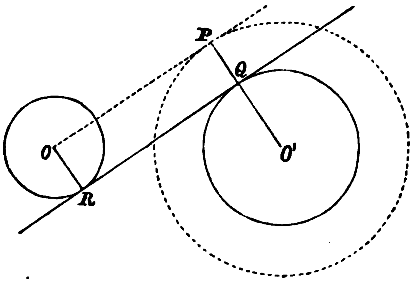
With centre O and radius OA or OB describe a circle, which will touch AD at A and pass through B, and therefore the segment AFB contains an angle equal to the angle BAD, that is to the given angle C.

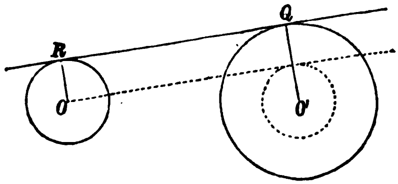
Problem 5.

To draw a common tangent to two given circles.

Let the centres of the circles be O, O'.

Construction. With centre O' and radius equal to the sum or difference of the radii of the given circles, describe a circle, as in the figures.





From O draw a tangent to this circle, touching it in P. Join $O'P$, and let it, produced through P if necessary, meet the circumference of the circle whose centre is O' in the point Q. Through O draw OR parallel to PQ, and join QR. QR will be a tangent to both circles.

Proof. Since PQ is by the construction equal and parallel to OR, therefore RQ is parallel to OP. But OP is at right angles to $O'P$, since it touches the circle in P, and therefore RQ is at right angles to OR and OQ; and therefore touches both circles.

COR. 1. *When the circles are wholly outside one another, they have four common tangents: when they touch externally, they have three common tangents: when they intersect one another, they have two common tangents: when they touch internally, they have one common tangent: and when one of the circles is wholly inside the other, they have no common tangent.*

COR. 2. *By symmetry with respect to OO', pairs of the common tangents will intersect on that line.*

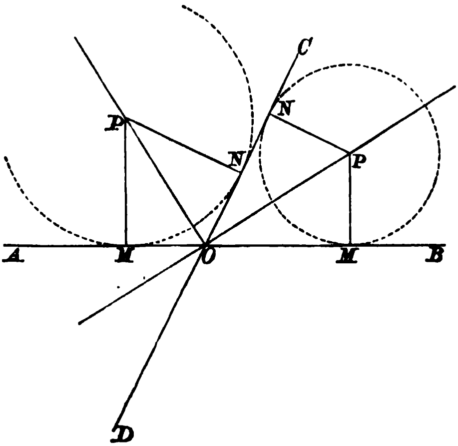
PROBLEM 6.

To find the locus of the centres of circles which touch two given straight lines.

Let AOB, COD be the two given straight lines, intersecting one another in O.

Let P be the centre of a circle which touches both the lines, PN, PM the perpendiculars from P on DOC and AOB.

Then $PN = PM$, and if OP be joined, since the triangles ONP, OMP are right angled at N and M, have the



hypothenuse OP common, and have one side $PN =$ one side PM, being radii of the same circle; therefore the triangles are equal in all respects, and the angle PON = the angle POM, that is OP bisects the angle COB.

Therefore the centres lie on the bisectors of the angles between the given lines; and these bisectors are therefore the locus required.

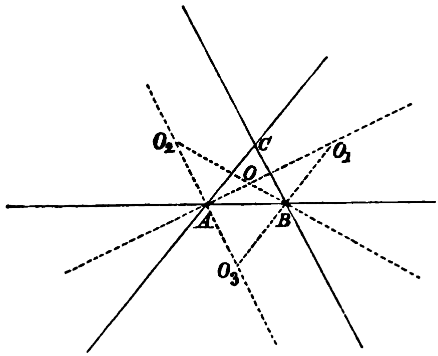
COR. 1. *It is obvious that these bisectors form two straight lines at right angles to one another.*

COR. 2. *If the given lines are parallel, the locus is a line parallel to both and equidistant from them.*

PROBLEM 7.

To describe a circle to touch three given straight lines of indefinite length.

Let the three given lines intersect in A, B, and C.



Then since the circle required is to touch the lines that intersect in A, its centre must lie on one of the bisectors of the angles at A. Similarly, it must lie on one of the bisectors of the angles at B. Therefore the construction is suggested.

Construction. Draw the bisectors of the angles at A and B, which will intersect in four points O, O_1, O_2, O_3.

These will be the centres of the circles required, and a circle described with any one of these points as centre, to touch one of the given lines, will touch the other two.

Cor. 1. *It follows that COO_3 and O_2CO_1 are straight lines, that is, the six bisectors of the interior and exterior angles of a triangle intersect one another three and three in four points.*

Cor. 2. *If two of the lines are parallel, only two circles can be described to touch the three lines.*

Cor. 3. *If all the lines are parallel, no circle can be described to touch them all.*

SECTION V.

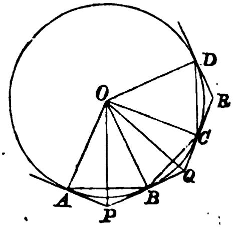
Regular Polygons.

One of the most interesting species of problem connected with the circle consists in describing regular polygons, and inscribing them in, or circumscribing them about given circles.

We shall first establish the following theorems.

Theorem 12.

If from the centre of a circle radii are drawn to make equal angles with one another consecutively all round, then if their extremities are joined consecutively, a regular polygon will be inscribed in the circle, and if at their extremities, tangents are drawn, a regular polygon will be circumscribed to the circle.



Let OA, OB, OC, OD ... be radii making equal angles consecutively to one another.

(1) Join AB, BC, CD ...

Then since the angles at O are equal, the chords which subtend them are equal (II. 1), and therefore the inscribed polygon is equilateral.

And since the arcs AB, BC, CD ... are equal; therefore the arc AC = arc BD, and therefore the angle ABC in the segment ABC = the angle BCD in the segment BCD; that is, the polygon is equiangular.

Hence $ABCD$ ... will be a regular polygon inscribed in the circle.

(2) Draw tangents at A, B, C, D ... meeting one another in P, Q, R ...

Join PO, QO.

Since $AP = PB$, the line PO bisects the angle AOB, and similarly QO bisects the angle BOC. Therefore the angle POB = the angle QOB, and hence $PB = BQ$ from the triangles POB, QOB.

Therefore QP is double of QB; and similarly QR is double of QC.

But $QB = QC$; and therefore $QP = QR$; and thus it may be shewn that the polygon is equilateral.

And since the angles APB, BQC ... are supplementary to the angles AOB, BOC ... they are equal to one another.

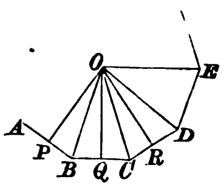
That is, the polygon is equiangular; and since PQ, QR ... are tangents, PQR ... will be a regular polygon circumscribed to the circle.

THEOREM 13.

In a regular polygon the bisectors of the angles intersect in one point, which is the centre of the circles inscribed in the polygon, or circumscribed about it.

Let $ABCDE$ ... be a regular polygon.

Bisect the angles B, C by straight lines meeting in O.



Join OD, OE ...

Since BO, CO are bisectors of the equal angles ABC, BCD, therefore the angle OBC = the angle OCB; and therefore $OB = OC$.

And because in the triangles OCB, OCD, the angle

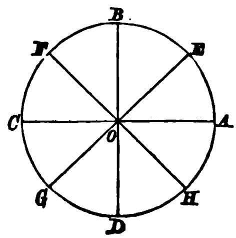
$OCB = OCD$, and the sides which contain these equal angles are equal to one another, each to each, therefore $OD = OB$, and $ODC = OBC$. But OBC is half of ABC, that is of CDE, since the polygon is equiangular. Therefore ODC is half of CDE, and therefore OD bisects the angle CDE.

In the same manner it may be shewn that $OE = OC$, and bisects the angle at E. Hence O is the centre of the circle circumscribed about the polygon, of which OB is the radius.

And because AB, BC, CD ... are equal chords in this circle, of which O is the centre, therefore the perpendiculars OP, OQ, OR on those chords are all equal; and a circle described with centre O and radius OP will pass through Q, R ... and touch AB, BC, CD ... in the points P, Q, R ... That is O is the centre of the inscribed circle of which OP is the radius.

Problem 8.

To construct a regular polygon of four, eight, sixteen ... sides, and inscribe them in, or describe them about a given circle.



Take O the centre of the given circle, and draw through it two diameters at right angles to one another, meeting the circumference in A, B, C, D; then by Theorem 12, if AB, BC ... be joined, we get a regular inscribed polygon of four sides; and if tangents be drawn at A, B, C and D, we get a regular circumscribed polygon of four sides, that is a square.

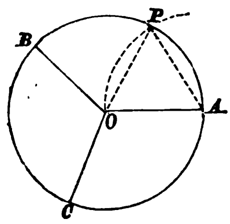
To describe an octagon, bisect (by Part I. Prob. 1) the angles AOB, BOC, COD, DOA by lines meeting the circle in E, F, G, H; then A, E, B, F... are the angular points of a regular inscribed octagon, or the points of contact of the sides of a regular circumscribed octagon.

Similarly, by again bisecting the angles at O, a regular sixteen-sided polygon may be constructed; and hence a thirty-two-sided figure, and so generally a polygon of 2^n sides, may be constructed, when n is any integer greater than 1.

PROBLEM 9.

To construct regular polygons of three, six, twelve ... sides, and inscribe them in, or describe them about a given circle.

Let O be the centre of the circle. On OA, one of the



radii, make an equilateral triangle AOP, and take AOB, double of the angle AOP.

Then since each angle of an equilateral triangle is one-third of two right angles; therefore its double is one-third of four right angles; and therefore two other equal angles BOC, COA will fill up the space round O.

Hence the angles at O being equal, A, B, C are the angular points of a regular inscribed polygon of three sides.

As before, by bisecting the angles at O we obtain the angular points of the regular hexagon; and by bisecting these angles we obtain the dodecagon; and so generally a regular polygon of 3×2^n sides may be constructed, where n may have any integral value, including zero.

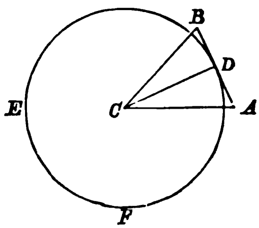
The Theorems at present proved do not enable the student to construct any regular polygons except those included in the foregoing problems.

Hereafter we shall shew that a regular pentagon can be described, and by means of it the decagon, quindecagon, &c.

THEOREM 14.

The area of a circle is equal to half the rectangle contained by the radius and a straight line equal to the circumference.

Let C be the centre of a circle DEF. And let AB be the side of a polygon described about the circle, CD a radius drawn to the point of contact.



Then the triangle $ABC = \frac{1}{2}$ the rectangle contained by CD and AB.

And since the whole polygon can be divided into triangles by lines drawn from the angular points to the centre,

The area of the polygon = $\frac{1}{2}$ the rectangle contained by the radius of the circle and perimeter of the polygon.

Now this is true whatever may be the number of sides of the polygon.

But by perpetually increasing the number of sides of the polygon, that polygon approaches nearer and nearer to the circle.

So finally the area of the circle = $\frac{1}{2}$ rectangle contained by radius and circumference.

Remark. This will give the area of the circle when we know the lengths of the radius and the circumference. The length of the latter cannot be found by the use of the rule and compasses. But if we added to our mathematical instruments a cylinder that might be rolled on the paper, or a tape that might be unrolled from a cylinder, we should have that length at once, and consequently the area.

MISCELLANEOUS THEOREMS AND PROBLEMS.

1. Prove that the two tangents drawn to a circle from any external point are equal.

2. If from a point without a circle two tangents *AB*, *AC* are drawn, the chord of contact *BC* will be bisected at right angles by the line from *A* to the centre.

3. If a circle is inscribed in a right-angled triangle, the excess of the two sides over the hypothenuse is equal to the diameter of the circle.

4. If a quadrilateral figure be described about a circle, the sums of the opposite sides will be equal to one another.

5. If a six-sided figure be circumscribed about a circle, the sums of the alternate sides will be equal.

6. If a quadrilateral figure be described about a circle, the angles subtended at the centre by any two opposite sides are together equal to two right angles.

7. *AOB*, *COD* are two chords of a circle at right angles to one another; prove that the squares of *OA*, *OB*, *OC*, and *OD*, are together equal to the square of the diameter.

8. The chord AB is produced both ways equally to C, D, and tangents CE, DF, drawn on opposite sides of CD; shew that EF bisects AB.

9. Two circles touch one another in A, and have a common tangent BC. Shew that the angle BAC is a right angle.

10. Describe (when possible) a circle of given radius to touch (1) two given lines, (2) two given circles.

11. Describe a circle to touch a given line in a given point, and pass through another given point.

12. Describe a circle to touch a given circle in a given point, and to pass through another given point.

13. Find the following loci:—the vertices of triangles on the same base, having a given vertical angle.

14. Of the point of intersection of the lines which bisect the angles at the base of such triangles.

(Prove that the angle between each pair of intersectors is the same.)

15. Of a point at which a given straight line subtends a given angle.

16. Of the points of bisection of parallel chords in a circle.

17. Of the points of bisection of equal chords in a circle.

18. Of points from which the tangent to a given circle has a given length.

19. Of the vertices of right-angled triangles on a given base.

20. Of the centres of all circles which touch a given line in a given point.

21. Of the centres of circles which touch a given circle in a given point.

22. Of the centres of circles of given radius which pass through a given point.

23. Of the middle point of a line drawn from a given point to meet a given circle.

24. Shew that the inscribed equilateral triangle is one-fourth of the circumscribed equilateral triangle.

25. A ladder slips down a wall: find the locus of its middle point.

26. If from two fixed points in the circumference of a circle two lines are drawn to intercept a given arc, the locus of their intersections is a circle.

27. Two chords of a circle which do not bisect each other do not pass through the centre.

28. Circles which cut one another cannot have the same centre.

29. Two shillings are moved in the corner of a box so that each always touches one side, and they touch one another; find the locus of the point of contact.

30. Two circles cut one another, and lines are drawn through the points of section, and terminated by the circumferences; shew that the chords which join the extremities of these lines are parallel.

31. Two equal circles intersect in A and B, and any line BCD is drawn to cut both circles. Prove that

$$AC = AD.$$

32. Two equal circles intersect in A, B; a third circle is drawn, with centre A and any radius less than AB, meeting the circles in C, D, on the same side of AB. Prove that B, C, D lie in one straight line.

33. ACD, ADB are two segments of circles on the same base AB; take any point C on the segment ACB, and join CA, CB, and produce them if necessary to meet ADB in D, E. Shew that the arc DE is constant.

34. If two circles cut each other, and from either point of intersection diameters be drawn, the extremities of these diameters and the other point of intersection shall be in the same straight line.

35. If a straight line be drawn to touch a circle and parallel to a chord, the point of contact will bisect the arc cut off by that chord.

36. Perpendiculars AD, CE are let fall from the angles A, C of the triangle ABC on the opposite sides. Prove that the angle ACE is equal to the angle ADE.

37. Two circles intersect in A, B, and tangents AC, AD are drawn to each circle, meeting circumferences in C, D, prove that BC, BD make equal angles with BA.

38. If one of two intersecting circles pass through the centre of the other, prove that the tangent to the first at the point of intersection, and the common chord, make equal angles with the radius to that point from the centre of the second.

39. Given base, altitude, and vertical angle, construct the triangle.

40. To draw a line from a given point such that the

perpendicular on it from a given point shall have a given length.

41. In a given straight line to find a point at which a given straight line subtends a given angle.

42. Describe a circle to touch a given circle, and touch a given line in a given point.

43. Describe a circle of given radius to touch a given line, and have its centre on another given line.

44. Find a point in a given chord produced of a circle, from which the tangent to the circle shall have a given length.

45. With a given radius describe a circle touching two given circles.

46. Describe a triangle, having given the vertical angle and the segments of the base made by the line bisecting the vertical angle.

47. Given base, altitude, and radius of circumscribed circle, construct the triangle.

48. The triangle contained by the two tangents to a circle from any point and any other tangent that meet them has its perimeter double of either of the two tangents. Prove this; and apply it to construct a triangle, having given the vertical angle, perimeter, and altitude.

49. Given the perimeter, the vertical angle, and the line bisecting the vertical angle, construct the triangle.

50. If two of the external angles of a triangle and one internal angle are bisected, prove that the three bisectors intersect in one point.

51. The three perpendiculars to the sides of a triangle drawn through their middle points meet in one point.

52. The three lines which join the angles of a triangle to the middle points of the opposite sides intersect in one point.

53. If two circles touch one another, the lines which join the extremities of parallel diameters towards opposite parts will intersect in the point of contact.

54. The circles described on the sides of a triangle as diameters intersect in the sides, or sides produced, of the triangle.

55. Equilateral triangles are described on the sides of a triangle; prove that the circles described about those triangles pass through one point.

56. If tangents be drawn at the extremities of any two diameters of a circle, the straight lines joining the opposite points of intersection will both pass through the centre.

57. The four common tangents to two circles which do not meet one another intersect, two and two, on the straight line which joins the centres of the circles.

58. Given the altitude, the bisector of the vertical angle, and the bisector of the base, to construct the triangle.

59. Draw circles to touch one side of a given triangle and the other two sides produced.

60. Draw a figure of a triangle with its inscribed, circumscribed, and escribed circles.

61. If a triangle is equilateral, shew that the radii of

the inscribed, and circumscribed, and an escribed circle are to one another as 1, 2, 3.

62. If circles are described with the vertices of a triangle as centres, and so as to pass through the points of contact of the inscribed circle with the adjacent sides, these three circles will touch one another.

63. Place a straight line of given length in a circle so that it shall be parallel to a given diameter of the circle.

64. Place (when possible) a straight line of given length in a circle so that it shall pass through a given point within or without the circle.

65. Given three points describe circles from them as centres so that each may touch the other two.

66. On the side of any triangle equilateral triangles are described externally, and their vertices joined to the opposite vertices of the given triangle, shew that the joining lines pass through one point.

67. O is the centre of the circle inscribed in the triangle ABC, which touches AB, AC in C', B'; if AO cuts the circle in P, and AO produced in P', shew that P, P' are the centres of the inscribed and escribed circles of the triangle $AB'C'$.

68. Shew that a triangle is equal to the rectangle contained by its semi-perimeter and the radius of the inscribed circle.

69. Of all the rectangles inscribable in a circle, shew that a square is the greatest.

70. If a quadrilateral figure can be described about a circle, shew that the sums of its opposite sides are equal. Can a circle be inscribed in (1) a rectangle, (2) a parallelogram, (3) a rhombus?

71. Shew that the inscribed hexagon is three-fourths of the circumscribed hexagon.

72. Shew that the six segments into which the points of contact of the escribed circles of a triangle divide the sides, may be arranged in three pairs of equal segments.

73. Inscribe an octagon in a given circle.

74. Describe a circle (1) to touch three given lines; (2) to intercept equal chords of any given length on three given straight lines.

In how many ways may each of these problems be solved?

75. At any point in the circumference of the circle circumscribing a square, shew that one of the sides subtends an angle three times as great as the others.

76. Find the locus of points at which two sides of a square subtend equal angles.

77. Find the locus of points at which three sides of a square subtend equal angles.

78. If four straight lines intersect one another so as to form four triangles, prove that the four circumscribing circles will pass through one point.

79. Of all triangles inscribable in a circle the greatest is the equilateral. Extend this to the case of a polygon of any number of sides.

80. A straight line is divided into any two parts in C, and ADC, CEB are equilateral triangles on the same side of AB. Find the locus of the intersection of AE and BD.

BOOK III. PROPORTION.

INTRODUCTION.

MEASURES.

A *measure* of a line is any line which is contained in it an exact number of times. Thus an inch is a measure of a foot; and a yard is a measure of a mile. So too the measure of an area is any area which is contained an exact number of times in it. A square inch is thus a measure of a square yard. *A measure is therefore an aliquot part of any magnitude which it measures.* The length of a line, the extent of an area, or any other magnitude, is completely known when we know a measure of it, and how many times it contains that measure.

In measuring any magnitude we take some standard to measure by. Thus in measuring length we take a yard, or a foot, or an inch. In measuring solids we take a cubic inch, a cubic foot, or the like. The standard so taken is called the *unit.* It may be a precise measure of the magnitude measured, or it may not. The number, whether whole or fractional, which expresses how many times a magnitude contains a certain unit is called the *numerical value* of that magnitude in terms of that unit. Thus in speaking of a

line as 7 yards long, a yard is the unit of length, and the numerical value of the line in terms of that unit is 7.

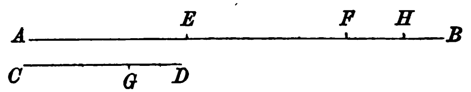
Two lines or magnitudes of the same kind are said to have *a common measure* when there exists a unit of which they can both be expressed as multiples. Thus 15 inches and 1 foot have a common measure, for with the unit 3 inches, their numerical values would be 5 and 4; and with the unit 1 inch their numerical values would be 15 and 12. All whole numbers have unity as a common measure.

The following problem gives a method of finding the greatest common measure of two magnitudes, if any common measure exists.

PROBLEM.

To find the greatest common measure of two magnitudes, if they have a common measure.

Let AB and CD be the two magnitudes. From AB



the greater cut off parts, AE, EF... each equal to CD the less, leaving a remainder FB which is less than CD.

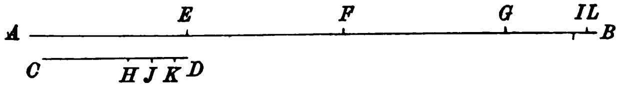
From CD cut off parts, CG..., equal to FB, leaving a remainder GD less than FB.

From FB cut off parts FH, HB... equal to GD: and continue this process until a remainder GD is found which is contained *an exact number* of times in the previous remainder, so that no further remainder is left. The last remainder is then the greatest common measure.

For, firstly, since GD measures FB, it also measures CG; and therefore measures CD. But $CD = AE$ and EF; and therefore GD measures AE, EF and FB; that is it measures AB. Hence GD is *a* common measure of AB and CD.

And again, since every measure of CD and AB must measure AF, it must measure FB or CG, and therefore also GD: hence the common measure cannot be greater than GD; that is GD is the *greatest* common measure.

So also, in the figure adjoining, the first remainder is



GB; the second HD; the third IB; the fourth KD, which is contained exactly twice in IB. Hence KD is the greatest common measure, and it will be seen to be contained twice in IB, and therefore five times in HD, seven times in GB, 12 times in CD, and 43 times in AB.

Hence AB and CD have as their numerical values 43 and 12 in terms of the unit KD.

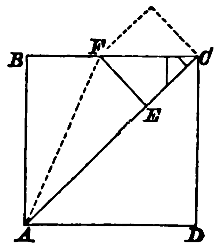
COR. Every measure of KD is a common measure of AB and CD.

When magnitudes have a common measure they are called *commensurable*. But it is very frequently the case in Geometrical figures, that lines and other magnitudes have no common measure; the process above given continuing indefinitely; the remainder becoming smaller at each step of the process but never actually disappearing. In this case the lines are said to be *incommensurable*.

The following theorem will serve to illustrate incommensurable magnitudes.

THEOREM I.

To prove that the side and diagonal of a square are incommensurable.



Let $ABCD$ be a square: AC the diagonal.

Then will AC and AB be incommensurable.

For since AC is $> AB$ and $<$ twice AB, cut off a part $AE = AB$. Then the common measure of AC and AB, is also a common measure of EC and AB, or of EC and BC.

Draw EF at right angles to AC to meet BC in F: and join AF. Then in the right-angled triangles ABF, AEF the hypothenuse and one side $AB =$ the hypothenuse and one side AE, and therefore $BF = FE$.

And in the triangle FEC, since E is a right angle, and ECF half a right angle, therefore EFC is half a right angle, and therefore $FE = EC$. Therefore $EC = BF$.

Hence the common measure of EC and BC is also a common measure of EC and FC.

But EC and FC are the side and diagonal of a square smaller than $ABCD$; and similarly the common measure of EC and FC can be shewn to be also the common measure of the side and diagonal of a still smaller square; and so on, till the side and diagonal become as small as we please.

Hence it is evident that we shall never find a common measure of AC and BC. For the original relation of side and diagonal is perpetually reproduced, and we are perpetually brought back to the problem with which we started.

The remainder becomes smaller at every step but never actually disappears. There is therefore no common measure of AC and BC; that is, they are incommensurable.

Ratio.

When two magnitudes of the same kind are considered we can compare them with one another, and form a notion of their relative magnitudes: we do this instinctively, and antecedently to all geometrical teaching. To express the notion thus formed we must ascertain how many times the one contains the other, or some aliquot part of the other. Thus if one line contains another 7 times, the one line has the same relative magnitude to the other that the number 7 has to the number 1; and if one line contains the 5th part of another 8 times, the one has to the other the same relative magnitude that 8 has to 5.

This relation of magnitude is called ratio.

It follows from the notion of relative magnitude that the ratio of two magnitudes is the same as that of their numerical values in terms of the same unit. And if two magnitudes have the same numerical values in terms of different units, their ratio is the same as that of their units.

Thus the ratios of the lines whose Greatest Common Measures were ascertained above, are the ratios of 12 to 5 and 43 to 12 respectively. Again the ratio of 7 weeks to 7 days is that of a week to a day.

The fact is that all number is but a kind of ratio.

We are taught in arithmetic to distinguish between *concrete* and *abstract quantities*. Concrete quantities are really existing things: abstract quantities are the means that we use to express the concrete. Seven shillings, five horses, three acres are concrete quantities: seven, five, three

are abstract quantities. The difference is most clearly seen in multiplication. You cannot multiply by a concrete quantity; a multiplier must be abstract. You cannot multiply seven shillings by six shillings, nor five days by three weeks, nor twelve miles by ten cats. Now *abstract quantities* and *ratios* are precisely the same things. We use the phrase "*abstract quantity*" when we are thinking of a quantity in itself; we use the word "*ratio*" when we are thinking of the relation of one quantity to another. The number seven considered in itself is called an abstract quantity, considered as expressing the relation of a week to a day, of a guinea to three shillings, of the number 42 to the number 6, it is called a ratio.

All numbers, therefore, as we said before, are ratios. But there is an important distinction between numbers and ratios; and though all numbers are ratios, all ratios are not numbers.

Numbers are essentially discontinuous, and therefore unsuited immediately to express the relations of magnitudes that change continuously. If a line 3 inches long is conceived as stretched till it becomes 4 inches long it passes continuously from one length to the other, but the arithmetical expression of the change is discontinuous, however small a unit we take: if we take $\frac{1}{1000}^{\text{th}}$ of an inch as the unit, the line is at first 3000, then 3001, 3002,...and finally 4000 of these units, but we cannot arithmetically express all the values the line takes in passing from 3000 to 3001; that is, all the ratios which its length has to $\frac{1}{1000}^{\text{th}}$ of an inch.

Here lies the fundamental difficulty in the application of arithmetic to Geometry. Arithmetic deals with numbers which are discontinuous. Geometry with ratios which are

continuous and only coincide with numbers at regular intervals. We may make those intervals as small as we please, but we cannot get rid of them.

Two commensurable magnitudes have to one another a ratio which can be expressed in numbers. For if they contain their common measure m times and n times respectively, they have the same ratio as m to n.

This is generally called the ratio of m to n, which is written $m : n$ or $\frac{m}{n} : 1$ or more simply $\frac{m}{n}$; and if A, C are the magnitudes it is said that $\frac{A}{C} = \frac{m}{n}$ or that $A = \frac{m}{n} C$.

Incommensurable magnitudes have to one another a ratio, but they have not to one another the ratio of any two numbers however great. For if they had the ratio of $p : q$, and the first were divided into p equal parts, the second would contain q of those parts; and thus they would have a common measure, viz. one of these parts.

But numbers can be found which express within any specified degree of accuracy the ratio of incommensurable magnitudes.

For if A and B be the magnitudes, and B is divided into n equal parts, there must be some number m, such that A contains m but not $m + 1$ such parts; therefore the ratio of $A : B$ is greater than $\frac{m}{n}$, but less than $\frac{m+1}{n}$; and these ratios differ only by $\frac{1}{n}^{\text{th}}$, which by increasing n may be made as small as we please.

This conclusion may be differently expressed thus; that if A and B are incommensurable, they may be made commensurable by adding to either of them a magnitude less

than one which shall be as small as we please. For by adding to A a quantity less than $\frac{1}{n}^{\text{th}}$ of B, A thus increased would contain the n^{th} part of B $\overline{m+1}$ times exactly.

It follows from what has been said that the ratio of two magnitudes is the quantity of one in terms of the other.

It is important to observe that since magnitude and ratio are both continuous, there is always a magnitude that has a given ratio to any given magnitude.

Compound Ratio.

When a magnitude is altered successively in two or more ratios it has to the final result a ratio which is compounded of the given ratios.

Thus if $m : n, p : q$ be two ratios, and a magnitude A is first altered in the ratio $m : n$, and then the result altered in the ratio $p : q$, and the final result thus obtained is B, then the ratio $A : B$ is compounded of the ratio $m : n$, $p : q$; for the single alteration indicated by ratio $A : B$ produces the joint effect of the two alterations indicated by the ratio $m : n$, $p : q$.

This process is called the composition of ratios. It is plainly not necessary to suppose that A has been actually altered till it has become B. It is enough if B is what A would have become if so altered.

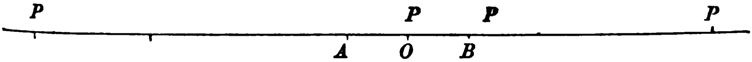
When two equal ratios are compounded the ratio resulting from the composition is said to be the duplicate of the original ratio. When three equal ratios, the triplicate; and so on.

It is evident that different ratios cannot have the same duplicate or triplicate ratio.

The following theorem furnishes a valuable exercise on ratios.

THEOREM 2.

If A *and* B *be two fixed points in a straight line of indefinite length, and* P *a moveable point in that line, then the ratio of* PA *to* PB *may have any value, from* 0 *to infinity, and there are two and only two positions of* P *such that* PA : PB = *any given ratio.*



The value infinity is represented by the symbol ∞.

Let O be the point of bisection of AB; then if P is at O the ratio $\frac{PA}{PB} = 1$.

Conceive the point P to move to the right towards B, then the ratio $\frac{PA}{PB}$ continually increases until, when P approaches indefinitely near to B the ratio becomes infinite; and for intermediate positions it has passed continuously through every value between 1 and ∞.

When P is at the right of B the ratio

$$\frac{PA}{PB} = \frac{PB + AB}{PB} = 1 + \frac{AB}{PB},$$

and is therefore greater than 1.

When PB is very small $\frac{AB}{PB}$ is very large, and as PB increases $\frac{AB}{PB}$ diminishes until it becomes indefinitely small,

and therefore $\frac{PA}{PB}$ becomes as nearly equal to 1 as we please, and has passed continuously through every value between ∞ and 1.

Hence for any assigned value of the ratio greater than 1 there are two positions for P, one between O and B, and one to the right of B.

Similarly as P moves from O to A, $\frac{PA}{PB}$ passes through every value from 1 to 0, and as it moves to the left of A it passes through every value from 0 to 1, and therefore for every value of the ratio less than 1 there are two positions for P, one between O and A, and one to the left of A.

PROPORTION.

Def. *Proportion* consists in the equality of ratios.

Four magnitudes are called *proportionals*, or are in proportion, when the 1st has the same ratio to the 2nd that the 3rd has to the 4th.

If A, B, C, D are the magnitudes in proportion, of which A and B are of the same kind, and C and D of the same kind, this is expressed by the notation $A : B :: C : D$, or by $\frac{A}{B} = \frac{C}{D}$. A and D are called the extremes, and B and C the means.

Def. The 1st and 3rd terms are called *homologous*, as occupying the same place as antecedent in the ratio; so also are the 2nd and 4th, as consequent.

The ratios of commensurable magnitudes may be expressed numerically by their numerical values in terms of

their common measure; and in such cases it is easy to ascertain whether the proportion is true. But if the magnitudes are incommensurable, their numerical values cannot be *exactly* expressed, and yet the ratios may be *exactly* equal, as is shewn in the following Theorem.

THEOREM 3.

If A, B, C, D *be four magnitudes such that* B *and* D *always contain the same aliquot part of* A *and* C *respectively the same number of times, however great the number of parts into which* A *and* C *are divided, then* A : B :: C : D.

For suppose A and B to be two magnitudes, and when

A is divided into n equal parts, let B contain m, but not $m+1$ of these parts. And let C and D be two other magnitudes, such that when C is divided into n equal parts, D contains m but not $m+1$ of these parts.

If D' is the 4th proportional to A, B, C, then D' also must contain m but not $(m+1)$ of these parts.

Now if D' differed from D by any quantity however small, it would be possible by sufficiently increasing n to make the n^{th} part of C still smaller than this quantity. And then when D contained this n^{th} part of C m times but not $m+1$ times, D' would contain it either less than m or more than $m+1$ times, according as D' was less or greater than D. But then D' would plainly not be the fourth proportional to A, B, and C. Therefore D' does not differ from D, that

is $D = D'$ and the four magnitudes A, B, C, D are proportionals.

This reasoning is frequently applicable in Geometry to prove that four magnitudes are proportionals, and is applicable alike to commensurable and incommensurable magnitudes.

Cor. 1. Permutando. *If* A, B, C, D *are magnitudes of the same kind, and* A : B :: C : D, *then is* A : C :: B : D.

For suppose A and B to be two magnitudes, and when

A is divided into n equal parts, let B contain m, but not $m+1$ of these parts. And let C and D be two other magnitudes, such that when C is divided into n equal parts, D contains m but not $m+1$ of these parts.

For let D' be the 4^{th} proportional to A, C, B, so that $A : C :: B : D'$. Then the ratio of A to C is that of the n^{th} part of A to the n^{th} part of C, into which parts they were divided: and since B must have to D' the same ratio, B and D' cannot contain a different number of these n^{th} parts respectively; so that since B contains m and not $m+1$ of the n^{th} parts of A, D' must contain m and not $m+1$ of the n^{th} parts of C. Now, precisely as before, if D' differed from D by any quantity however small, it might be shewn, by increasing n, to contain the n^{th} part of C either $m-1$ or $m+1$ times, which is impossible. Therefore $D' = D$ and $A : C :: B : D$.

Cor. 2. Invertendo. If A, B, C, D are in proportion

$$B : A :: D : C.$$

COR. 3. Componendo. Also $A+B : B :: C+D :: D$; and Dividendo. $A-B : B :: C-D : D$.

COR. 4. Ex æquali. If $A : B :: C : D$ and $B : E :: D : F$; then $A : E :: C : F$.

For $A : E$ is compounded of $A : B$ and $B : E$, and $C : F$ is compounded of the ratio $C : D$ and $D : F$ which are respectively equal to the ratios $A : B$ and $B : E$.

COR. 5. Addendo. If $A : B :: A' : B' :: A'' : B''$;

then $\quad A+A'+A'' : B+B'+B'' :: A : B$.

This follows from Cor. 1 and Cor. 3.

Remark. All these derived proportions may be obtained by the following method:—

(1) If A and B are commensurable, and therefore also C and D, let them have as their numerical values a, b, c, d respectively.

Then since $\quad A : B :: C : D$,

therefore $\quad \dfrac{a}{b} = \dfrac{c}{d}$ and $\therefore \dfrac{a}{c} = \dfrac{b}{d}$,

and therefore $\quad A : C :: B : D$.

(2) If A and B are incommensurable, and therefore also C and D, let B' and D' be taken differing from B and D by quantities less than $\dfrac{1}{n}^{\text{th}}$ of A and $\dfrac{1}{n}^{\text{th}}$ of C respectively (see p. 132), so that $A : B' :: C : D'$.

Then by the first case $A : C :: B' : D'$.

And since B' and D' differ from B and D by quantities which may be made as small as we please, we infer that

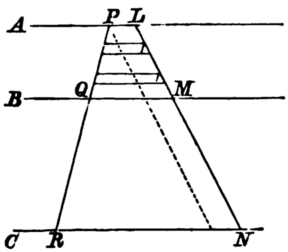
$$A : C :: B : D.$$

SECTION I.

APPLICATION OF PROPORTION TO LINES.

THEOREM 4.

If two straight lines are cut by three parallel straight lines, the segments made on the one are in the same ratio as the segments made on the other.



Let A, B, C be the three parallels, PQR, LMN any two lines intersected by them; then shall

$$PQ : QR :: LM : MN.$$

For if PQ be divided into any number of equal parts, and through the points of division lines be drawn parallel to A or B, LM will be divided by those lines into the same

number of equal parts, as may be proved by the methods of Book I. And if from QR parts equal to those of PQ are cut off, and lines drawn through the points of division, parallel to B or C, it is clear that MN will contain the same number of parts each equal to those of LM, that QR contains of those of PQ; that is, QR and MN will contain the same aliquot parts of PQ and LM the same number of times, however great the number of parts into which PQ and LM are divided.

Hence $\qquad PQ : QR :: LM : MN.$

COR. 1. It follows from the same reasoning that

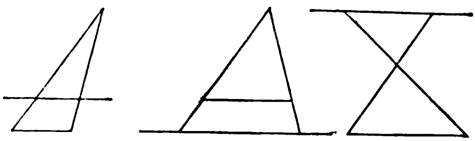
$$PQ : PR :: LM : LN,$$

and that $\qquad QR : PR :: MN : LN,$

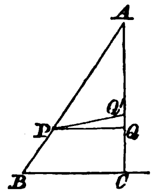
and that $\qquad PQ : LM :: QR : MN,$

(by TH. 3. CORS. 1, 2, 3).

COR. 2. *If a line be drawn parallel to one side of a triangle, it will cut the other sides or the other sides produced proportionally.*



COR. 3. The converse of this corollary is true. That is *if a line cuts two sides of a triangle, both internally or both externally, proportionally, the line shall be parallel to the base of the triangle.*



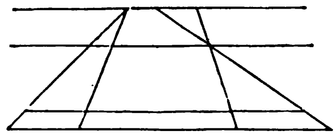
For let $AP : PB :: AQ : QC$, and suppose PQ not parallel to BC, but let PQ' be parallel if possible;

then $AP : PB :: AQ' : Q'C$,

and therefore $AQ : QC :: AQ' : Q'C$,

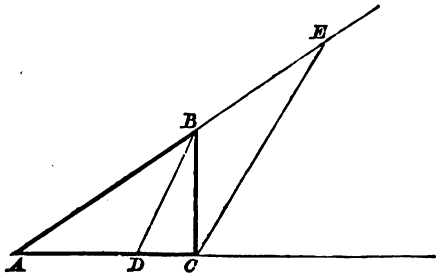
which is impossible.

COR. 4. *If any number of lines be parallel and be cut by other lines, the intercepts made on the latter by the parallels will be proportionals.*



THEOREM 5.

If a line bisect the vertical angle of a triangle and meet the base, it will divide the base into two segments which have to one another the ratio of the sides of the triangle.



Let ABC be a triangle, BD the bisector of the angle ABC.

Then will $AD : DC :: AB : BC$.

Draw CE parallel to BD to meet AB produced.

Then by parallelism the angle $BCE =$ the angle DBC, and the angle $BEC =$ the angle ABD. But $ABD = DBC$, and therefore the angle $BCE =$ the angle BEC; and therefore $BE = BC$.

But because AE, AC are cut by the parallels DB, CE;

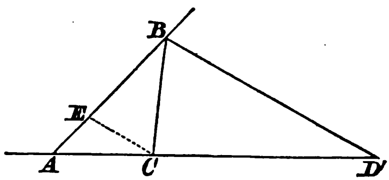
therefore $AD : DC :: AB : BE$,

that is, $AD : DC :: AB : BC$.

COR. 1. *Conversely, if* $AD : DC :: AB : BC$, *then* BD *is the bisector of the angle* ABC.

For there is only one internal bisector of the angle, and only one point D which divides the base internally, so that

$$AD : DC :: AB : BC.$$



COR. 2. *If* BD' *bisects the exterior angle,* D', *then also*

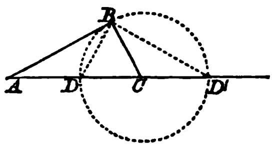
$$AD' : D'C :: AB : BC.$$

Draw CE parallel to BD', as before: and apply the same method of proof.

Cor. 3. *If* AB = BC, *then the ratio of* AD′ : D′C *becomes* = 1, *which indicates that* D′ *is at an infinite distance (by Theorem 2). Hence the external bisector of the vertical angle of an isosceles triangle is parallel to the base.*

Cor. 4. *If* B *moves so that the ratio* AB : BC *is constant, the bisectors of the interior and exterior angles will always pass through the fixed points* D, D′ *which divide* AC *internally and externally in that ratio.*

Cor. 5. *Hence the locus of* B *is a circle described on* DD′ *as diameter. For the angle* DBD′ *is a right angle; and therefore the semicircle on* DD′ *will always pass through* B.



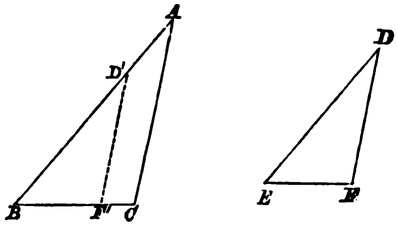
Def. Similar figures are such that the angles of the one are respectively equal to the angles of the other, and have the sides about the equal angles proportionals.

Theorem 6.

If two triangles have two angles of the one equal respectively to two angles of the other, the triangles shall be similar, the sides which are opposite the equal angles being homologous.

Let *ABC*, *DEF* be the two triangles, which have two

angles of the one equal to two angles of the other, and therefore have also their remaining angles equal.



Then shall they be similar, that is

$$AB : BC :: DE : EF,$$

and $$BC : CA :: EF : FD,$$

and $$CA : AB :: FD : DE.$$

Conceive the angle E placed on the angle B; then F and D would fall as F' and D' on BC and BA, or on those lines produced: and because the $\angle F =$ the $\angle C$, therefore $F'D'$ is parallel to CA;

and therefore $BF' : BC :: BD' : BA$,

and therefore $BF' : BD' :: BC : BA$,

that is $EF : ED :: BC : BA$.

Similarly by placing F on C, and D on A, the other proportions are obtained; and therefore the triangles are similar.

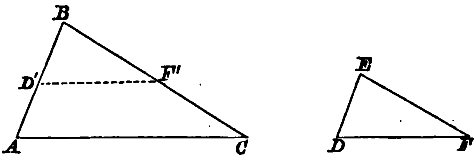
This theorem is a generalization of Theorem 15 in Book I. *If two angles and a side of one triangle are respectively equal to two angles and the corresponding side of another triangle, these triangles will be equal in all respects.*

Hence it will be observed that the equality of the angles involves the similarity of the triangles; and the additional equality of a pair of corresponding sides involves the identity of the triangles.

Theorem 7.

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, then will the triangles be similar.

Let the triangles ABC, DEF have the angles at B and



E equal, and let $BA : BC :: ED : EF$, then will the triangles be similar.

Conceive the angle E placed on the equal angle B, then D and F will fall as at D' and F' on the sides BA, BC,

and since $\quad BA : BC :: ED : EF$,

therefore $\quad BA : BD' :: BC : BF'$,

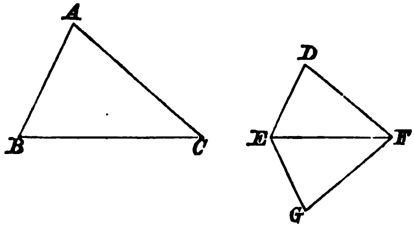
and therefore $D'F'$ is parallel to AC, Th. 4. Cor. 3.
and the angles $BD'F'$ and $BF'D'$, that is, D and F, are equal respectively to the angles A and C. Hence the triangles are equiangular and therefore similar.

It will be observed that this theorem is a generalization of Book I. Theorem 16. *If two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another, the triangles will be equal in all respects.*

THEOREM 8.

If the sides about each of the angles of two triangles are proportionals, the triangles will be similar.

Let ABC, DEF be two triangles which have their sides



about each of their angles proportional,

that is, $AB : BC :: DE : EF$,

and $BC : CA :: EF : FD$,

and $CA : AB :: FD : DE$.

Conceive a triangle equiangular to ABC applied to EF, on the opposite side of the base EF, so that the angles FEG, EFG are equal to B and C respectively.

Then the triangle GEF is equiangular to ABC, and therefore similar to it,

and therefore $GE : EF :: AB : BC$,

but $AB : BC :: DE : EF$,

and therefore $GE : EF :: DE : EF$,

and therefore $GE = ED$.

Similarly $GF = DF$,

and the triangle DEF is therefore equiangular to GEF, and therefore also to ABC.

Therefore the triangle DEF is similar to the triangle ABC.

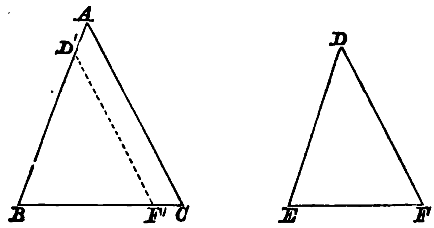
This theorem is a generalization of Book I. Theorem 18. *If the three sides of one triangle are respectively equal to the three sides of another, these triangles will be equal in all respects.*

Theorem 9.

If two triangles have the sides about an angle of the one triangle proportional to the sides about an angle of the other, and have also the angle opposite that which is not the less of the two sides of the one equal to the corresponding angle of the other, these triangles will be similar.

Let ABC, DEF be the two triangles, in which

$$BA : AC :: ED : DF,$$



and let AC be not less than AB, and DF therefore not less than DE, and also let the angle $B =$ the angle E.
Then shall the triangles be similar.

Cut off $BD' = ED$, and draw to BC an oblique $D'F'$ parallel to AC.

Then by similar triangles $BD'F'$ and BAC,

$$BD' : D'F' :: BA : AC,$$

that is, $$ED : D'F' :: BA : AC,$$

but $$ED : DF :: BA : AC,$$

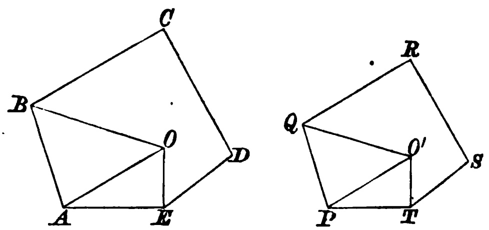
therefore $$DF = D'F',$$

and since DF is not less than DE, the two triangles $BD'F'$, EDF are equal in all respects by Book I. Theorem 19, that is, the triangle EDF is equiangular to the triangle $BD'F'$, and therefore also the triangle EDF is equiangular to ABC, and therefore similar to ABC.

This theorem is a generalization of Book I. Theorem 20.

THEOREM 10.

Similar polygons can be divided into the same number of similar triangles.



Let $ABCDE$, $PQRST$ be similar polygons, that is, let the angles $A, B, C\ldots$ of the one equal the angles $P, Q, R\ldots$ of the other respectively, and let the sides which contain the equal angles be proportional.

Then the triangles CAD, BAC have two angles CAD and CDA of the one equal respectively to BAC, BCA of the other; therefore they are equiangular, and similar.

In the same manner DCB is equiangular and similar to either DAC or CAB.

Cor. $AD : DC :: DC : DB$,

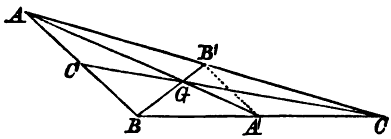
or the perpendicular from the right angle of a right-angled triangle on the hypothenuse is a mean proportional between the segments of the base.

Also $BA : AC :: AC : AD$,

and $AB : BC :: BC : BD$,

or the side of a right-angled triangle is a mean proportional between the hypothenuse and the projection on it of that side.

Ex. 2. *The three lines drawn from the angular points of a triangle to bisect the opposite sides will intersect in one point.*



Let ABC be the triangle, A', B', C' the middle points of its sides. Then AA', BB', CC' will pass through one point.

Join $B'A'$. Then since CA, CB are bisected in B', A',

$$\therefore\ CB' : B'A :: CA' : A'B,$$

and therefore $B'A'$ is parallel to AB,

and because the triangles CAB, $CB'A'$ are similar,

therefore $\quad AB : B'A' :: AC : B'C$;

but $AC = 2B'C$; and therefore $AB = 2B'A'$.

Now let AA' and BB' cut in G.

Then the triangles GAB, $GA'B'$ are equiangular,

and therefore $\quad AG : GA' :: AB : B'A'$;

but $\quad AB = 2B'A'$, and therefore $AG = 2GA'$,

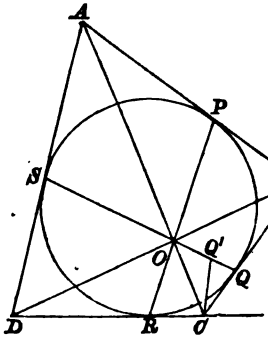
or $\quad GA' = \frac{1}{3}AA'$.

In the same manner it may be shewn that CC' cuts AA' at a point distant $\frac{1}{3}AA'$ from A'; that is, CC' passes through G the intersection of AA' and BB'.

Ex. 3. *If a quadrilateral figure be described about a circle, and the points of contact of opposite sides be joined, prove that these lines and the diagonals of the quadrilateral figure all intersect in one point.*

Let $ABCD$ circumscribe the circle, P, Q, R, S being the points of contact. Let AC cut SQ in O, draw CQ' parallel to AS.

Since SQ is the chord of contact of the tangents AS, BQ, the angle ASO = the angle BQO.



Therefore CQO is supplementary to ASO, or $OQ'C$, and therefore $CQO = CQ'Q$, and $CQ = CQ'$.

But from the triangles ASO, COQ', which are similar, we get

$$AO : CO :: AS : CQ' \text{ or } AS : CQ.$$

Similarly, if PR intersected AC in O', $AO' : CO' :: AP : CR$; but $AP : CR :: AS : CQ$, and therefore $AO : CO :: AO' : CO'$, therefore SQ and PR divide AC in the same ratio, or O and O' are the same points. Hence AC passes through the intersection of SQ and PR.

Similarly BD passes through the intersection of SQ and PR;

therefore the four lines AC, BD, PR, SQ pass through one point.

SECTION II.

Areas.

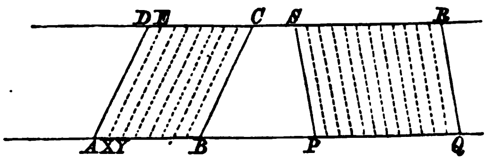
We have seen how *lines* are represented by *numbers;* a line being known when it contains a known unit a known number of times. We now proceed to the consideration of *areas* and their numerical representation.

The fundamental theorem is the following:

Theorem II.

Parallelograms of the same altitude are to one another as their bases.

Let $ABCD$, $PQRS$ be parallelograms of the same alti-



tude on the bases AB, PQ.

Then shall $DABC$ be to $SPQR$ as AB to PQ.

Divide AB into any number of parts AX, XY... and from PQ cut off parts equal to the parts of AB; and through X, Y... the points of section let lines be drawn parallel to the sides of the parallelogram.

Then the parallelograms DX, EY... into which $DABC$ is divided are all equal to one another, and equal to those cut off from the parallelogram $SPQR$, since they are on equal bases and of the same altitude; and therefore DX is the same aliquot part of $DABC$ that AX is of AB.

Hence the parallelogram $SPQR$ contains any aliquot part of $DABC$ as many times as PQ contains the same aliquot part of AB, into however many parts AB and DB are divided.

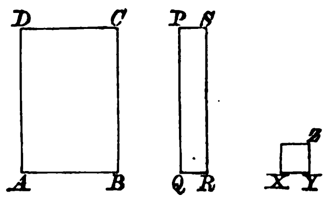
Therefore $AB : PQ :: DABC : SPQR$.

Cor. 1. *Triangles of the same altitude are to one another as their bases.*

For a triangle is half the parallelogram on the same base and having the same altitude as the triangle.

Cor. 2. *If the unit of area be the square on the unit of length, then will the numerical value of a rectangle be the product of the numerical values of its base and altitude.*

For let $ABCD$ be a rectangle, XY the unit of length,



and XZ the square on XY the unit of area. And let AB contain XY m times, and AD contain it n times.

Then will the numerical value of $ABCD$ be mn.

For construct a rectangle $PQRS$ on a base equal to XY, and with altitude equal to AD.

Then $\quad DABC : PQRS :: AB : QR,$

$$:: m : 1,$$

that is, $\quad DABC = m \,.\, PQRS,$

and $\quad PQRS : ZX :: PQ : XY,$

$$:: n : 1,$$

that is, $\quad PQRS = n \,.\, ZX;$

and therefore $\quad DABC = mn \,.\, ZX,$

that is, if ZX is the unit, the numerical value of $DABC$ will be mn.

This is generally expressed by saying that the area of a rectangle is the product of its base and altitude.

Remark. If the rectangle is a square, whose side is m, the area will be m^2: hence if AB is a line, or its numerical value, AB^2 represents either the geometrical square on the line, or the arithmetical square of the number which is the value of AB. And the rectangle contained by AB and CD may be written $AB \times CD$; for the product of the numbers AB and CD is the number which represents the rectangle, on the understanding that the unit of area is the square of the unit of length.

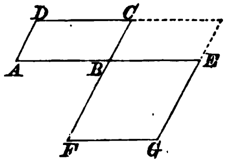
COR. 3. *The area of any parallelogram is the product of its base into its altitude.*

COR. 4. *The area of any triangle is half the product of its base and its altitude.*

COR. 5. *Hence the area of any rectilineal figure can be found by dividing it into triangles, and finding the areas of the triangles.*

THEOREM 12.

Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides.



Let $ABCD$, $EBFG$ be the parallelograms, and let them be placed so as to have AB, BE in one straight line, and therefore also, since the parallelograms are equiangular, so as to have CB, BF in one straight line.

Complete the parallelogram CBE.

Then the ratio of $DB : BG$ is compounded of the ratios of $DB : CE$ and of CE to BG.

But $\qquad DB : CE :: AB : BE,$

and $\qquad CE : BG :: CB : BF;$

therefore, the ratio of $DB : BG$ is compounded of the ratios $AB : BE$ and $CB : BF$.

COR. 1. *Triangles which have one angle of the one equal to one angle of the other, are to one another in the ratio compounded of the ratios of the sides containing that angle.*

COR. 2. *Equal parallelograms which are also equiangular, have their sides reciprocally proportional.*

For, in the figure above, the ratio compounded of the ratio $AB : BE$ and $CB : BF$ must be unity; and therefore $AB : BE :: BF : CB$, or the sides of the parallelograms are reciprocally proportional.

COR. 3. *Conversely, equiangular parallelograms which have their sides reciprocally proportional, are equal.*

COR. 4. *Hence, if four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means, and conversely.*

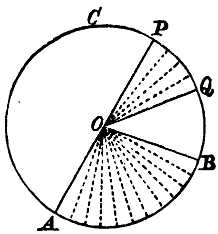
COR. 5. *If three straight lines are in continued proportion, that is, if the first is to the second, as the second to the third, then will the rectangle contained by the extremes be equal to the square on the mean, and conversely.*

COR. 6. *The ratio of two parallelograms or two triangles is the ratio compounded of the ratios of their bases and altitudes.*

THEOREM 13.

In any circle angles at the centre have to one another the ratio of the arcs on which they stand, or of the sectors which they include.

Let ABC be a circle, of which O is the centre.



And let AOB, POQ be two angles at the centre.

Then $\angle AOB : \angle POQ :: \text{arc}\, AB : \text{arc}\, PQ,$

$:: \text{sector}\, AOB : \text{sector}\, POQ.$

Divide the arc AB into any number of equal parts, and from the arc PQ cut off arcs equal to these. Join the points of division to O.

Then, since equal arcs subtend equal angles at the centre, the arc AB, and the angle AOB, are divided into the same number of parts.

And therefore POQ contains an aliquot part of AOB the same number of times that PQ contains the same aliquot part of AB, into however many parts AB and AOB are divided; therefore

$$\angle AOB : \angle POQ :: \text{arc}\, AB : \text{arc}\, PQ.$$

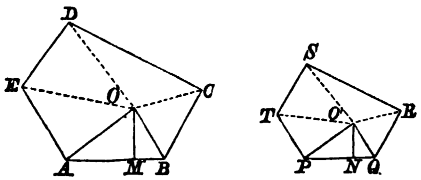
In precisely the same manner, since equal angles contain equal sectors, it may be shewn that

$$\angle AOB : \angle POQ :: \text{sector}\, AOB : \text{sector}\, POQ.$$

Theorem 14.

Similar polygons are to one another in the ratio of the squares of their homologous sides.

Let $ABCDE$, $PQRST$ be similar polygons.



Divide each of them into the same number of similar triangles by lines drawn from the points O, O'.

Let OAB, $O'PQ$ be two similar triangles.

Draw OM, $O'N$ perpendicular to AB, PQ.

Now $$\frac{OAB}{O'PQ} = \frac{AB \times OM}{PQ \times O'N} = \frac{AB}{PQ} \times \frac{OM}{O'N};$$

but by similar triangles OAM, $O'PN$ and OAB, $O'PQ$ we have

$$\frac{OM}{O'N} = \frac{OA}{O'P} = \frac{AB}{PQ}.$$

Therefore, substituting $\frac{AB}{PQ}$ for $\frac{OM}{O'N}$ above, we have

$$\frac{OAB}{O'PQ} = \frac{AB}{PQ} \times \frac{AB}{PQ}$$
$$= \frac{AB^2}{PQ^2}.$$

In the same manner it may be shewn that

$$\frac{OAE}{O'PT} = \frac{AE^2}{TP^2} = \frac{AB^2}{PQ^2},$$

since the polygons are similar, and therefore $\frac{AE}{TP} = \frac{AB}{PQ}$: and similarly for the other triangles;

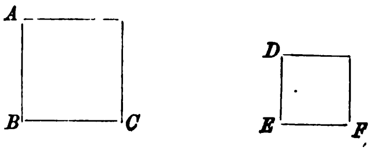
$$\therefore \frac{OAB}{O'PQ} = \frac{OAE}{O'PT} = \dots \qquad = \frac{AB^2}{PQ^2}.$$

Therefore the sum of OAB, OAE... is to the sum of $O'PQ$, $O'PT$... as $AB^2 : PQ^2$; that is, the polygons are to one another as the squares of their homologous sides.

COR. 1. *Hence the polygons are to one another as the squares of any homologous lines in them.*

The polygons are said to be *similarly described* on their homologous sides.

Cor. 2. It follows from (Theorem 12) that *the ratio of the squares on two lines is the duplicate ratio of the lines.* For if ABC, DEF are squares, then $AC : DF$ is the ratio



compounded of the ratios $AB : DE$ and $BC : EF$, that is, of $BC : EF$, and $BC : EF$; which is defined to be the duplicate ratio of $BC : EF$.

Moreover, if three straight lines are in continued proportion, the 1st : 3rd in the duplicate ratio of the 1st : 2nd.

Cor. 3. *Therefore further if three straight lines be in continued proportion, the 1st : 3rd as any polygon described on the 1st : the similar and similarly described polygon on the 2nd.*

Relation of Algebra to Geometry.

We now begin to see how algebra and arithmetic may assist Geometry. For example, we have learnt in algebra that $a^2 - b^2 = (a + b)(a - b)$. Now suppose a and b to be *the numerical values of two lines*, then $a + b$ is the numerical value of the sum of the line, $a - b$ is that of their difference.

And therefore $(a+b)(a-b)$ is the numerical value of the rectangle contained by the sum and difference of the two lines.

And a^2-b^2 is the difference of the numerical values of the squares on the two lines. Hence the algebraical identity $a^2-b^2=(a+b)(a-b)$ proves the geometrical fact, (which indeed we have learnt before) that the difference of the squares of two lines is equal to the rectangle contained by the sum and difference of those lines.

So again $(a+b)^2=a^2+2ab+b^2$ has a geometrical signification which the student should work out for himself in the same manner.

It will be useful for him to work out the geometrical meaning (with figures) of the following identities:

$$(a-b)^2=a^2-2ab+b^2,$$

$$a(a+b)=a^2+ab,$$

$$(a+b)^2=a(a+b)+b(a+b),$$

$$(a+b)^2+(a-b)^2=2a^2+2b^2,$$

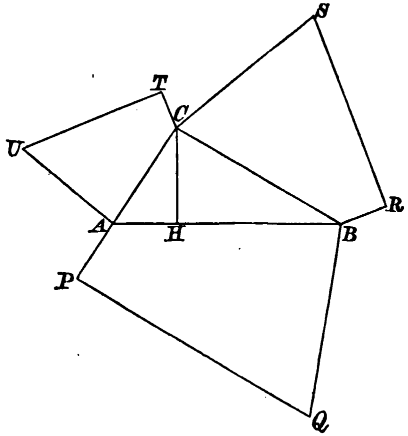
$$(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc.$$

The following theorems are useful deductions from the preceding.

EXAMPLES.

(1) *In any right-angled triangle any figure described on the hypothenuse is equal to the similar and similarly described figures on the two sides.*

Let ABC be a triangle right-angled at C; and let



$APQB$, $BRSC$, $CTUA$ be similar figures similarly described on the sides AB, BC, CA, that is, figures of which AB, BC, CA are homologous sides.

Then will $APQB = BRSC + CTUA$.

Draw CH perpendicular to AB.

Then $AB : BC :: BC : BH$ by similar triangles, and therefore

$APQB : BRSC :: AB : BH$ (Theorem 14, Cor. 3),

in the same manner it may be shewn that

$$APQB : CTUA :: AB : AH,$$

and therefore

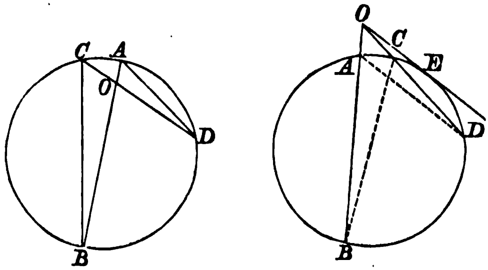
$$APQB : BRSC + CTUA :: AB : BH + AH;$$

but $$AB = BH + AH;$$

and therefore $$APQB = QRSC + CTUA.$$

It is obvious that a special case of this theorem is the theorem proved before, that the square on the hypothenuse of a right-angled triangle is equal to the sum of the squares on the sides.

(2) *If through any point* O *within or without a circle, chords are drawn, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.*



Let AOB, COD be the chords through O. Then is $AO \times OB = CO \times OD$.

For join CB, AD. Then since the angle D = angle B in the same segment, and the angle at O common to the two triangles AOD, BOC, the triangles are equiangular and similar,

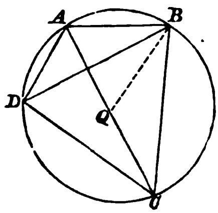
$$\therefore AO : OD :: CO : OB,$$
$$\therefore AO \times OB = CO \times OD.$$

COR. If one of the secants OCD, in figure 2, became a tangent, as OE, then OC and OD are equal to OE; and therefore $AO : OE :: OE : OB$, and $OE^2 = AO \times OB$, or *the square on the tangent to a circle from any point is equal to the rectangle contained by the intercepts on the secant drawn from that point.*

(3) *In a quadrilateral inscribed in a circle the rectangle contained by the diagonals is equal to the sum of the rectangles contained by the opposite sides.*

Let $ABCD$ be the quadrilateral inscribed in a circle. Then will

$AC \times BD = AB \times DC + AD \times BC.$



At the point B make the angle CBQ equal to the angle ABD.

Then in the triangles ABD, QBC, since the angle $QBC =$ the angle ABD, and the angle ADB = the angle QCB, therefore the triangles are similar; and therefore $AD : DB :: QC : BC$, and therefore $AD \times BC = DB \times QC$.

In the same manner, since the triangles ABQ, DBC are similar, $AB : QA :: DB : DC$, and therefore

$$AB \times DC = DB \times QA.$$

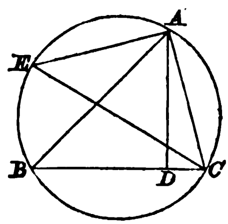
Therefore $AD \times BC + AB \times DC = DB \times QC + DB \times QA$
$$= DB \times AC.$$

This is known as Ptolemy's theorem.

(4) *In every triangle* ABC, *the rectangle contained by the two sides* AB, AC *is equal to the rectangle contained by the diameter* CE *of the circumscribing circle, and the perpendicular* AD, *let fall on* BC.

For the triangles ADB, CAE have the angles $ABD = CEA$ and $ADB = CAE$; therefore they are similar, and therefore

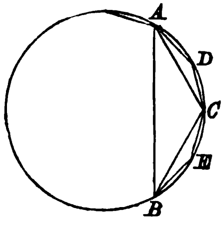
$$AB : AD :: CE : AC,$$
$$\therefore AB \times AC = AD \times CE.$$

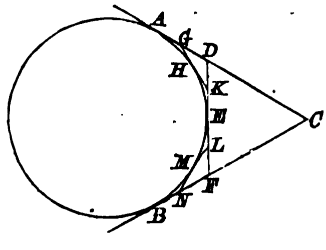


THEOREM 15.

Circumferences of circles have to one another the ratio of their radii.

The circumference of a circle is always greater than the perimeter of a polygon inscribed in it, less than that of a polygon circumscribed about it; and each polygon may be brought perpetually nearer to the circle by increasing the number of its sides, and ultimately the two polygons and the circle will coincide.

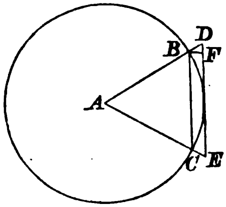




Thus in the left-hand figure the chord AB is less than AC, CB; and these are less than AD, DC, CE, FB; and these again, though more nearly coinciding with the arc AB, are less than that arc. Again, AC, CB in the right-

hand figure are greater than AD, DF, FB, and these again greater than AG, GK, KL, LN, NB, and these last, though more nearly coinciding with the arc AB, are greater than that arc.

Further, if we have two similar and regular polygons described inside and outside a circle, the difference between their perimeters may be as small as we please.



For if BC, DE be homologous sides of two such polygons, the difference between them will be DF, if BF be drawn parallel to AC. And we shall have

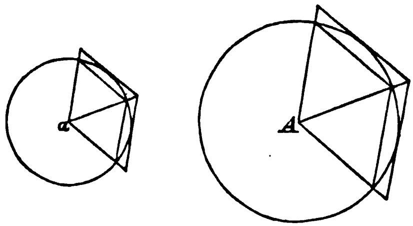
$$DF : DE :: DB : DA.$$

And this being true of all the sides is true of their sum, and therefore difference of perimeters : outer perimeter :: $DB : DA$.

Now $DB : DA$ may be made as small as we please, therefore the ratio of the difference of the perimeters to the outer perimeter may be made as small as we please; that is, since the outer perimeter does not increase but diminish at the same time, we may make the difference of perimeters as small as we please.

Now let c, C be the circumferences of two circles; r, R their radii; i, I the perimeters of two regular inscribed polygons, similar to each other; o, O the perimeters of two

regular circumscribed polygons, similar to each other and to the inscribed;



Then since the triangles in all the polygons are similar we shall have

$$r : R :: i : I,$$

and $$r : R :: o : O.$$

Now let $$r : R :: c : C'.$$

Then $$i : I :: c : C',$$

and since c is greater than i, C' is greater than I, so $o : O :: c : C'$, and since C is less than O therefore C' is less than O.

Therefore C' lies between I and O.

Now if C' differed from C by ever so small a quantity, we might have the difference between I and O still less, and then C would no longer lie between I and O, which is impossible.

So C' does not differ from C, and we have

$$r : R :: c : C.$$

Cor. Let a, A be the areas of the circles;

$$\therefore a : A :: rc : RC, \text{ (II. 14)}$$

$$:: r^2 : R^2,$$

that is, circles have to each other the ratio of the squares of their radii.

SECTION III.

PROBLEMS.

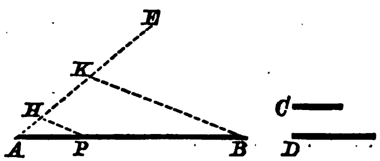
PROBLEM I.

To divide a given straight line into two parts which shall be in a given ratio.

Note. By a *given* ratio is meant the ratio of two given lines, or of two given numbers: and since two lines can always be found which have the ratio of two given numbers, it follows that a given ratio can always be represented by the ratio of two given lines.

Let AB be the given line, C and D the lines which have the given ratio; then it is required to divide AB into two parts, which have to one another the ratio of $C : D$.

Construction. From A draw a line AE making any angle with AB, and cut off parts AH, HK equal to C and



D respectively. Join KB, and draw HP parallel to KB. P will be the point of division required.

Proof. For since HP is parallel to KB,

therefore $AP : PB :: AH : HK$;

but $AH = C$; and $HK = D$;

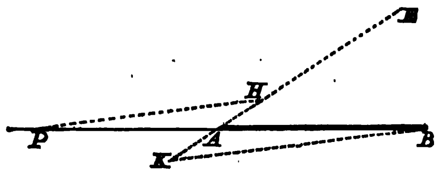
therefore $AP : PB :: C : D$,

that is, AB is divided into two parts which are to one another in the given ratio.

COR. 1. *In the same manner a line may be divided into any number of parts which have to one another given ratios.*

COR. 2. *Hence a line may be divided into any number of equal parts, the given ratios being all ratios of equality.*

Note. This construction divides the line *internally* into parts which have the given ratio. If it is required to divide



it *externally*, HK must be measured in the opposite direction along AE, as in the figure.

The proof will be the same as before.

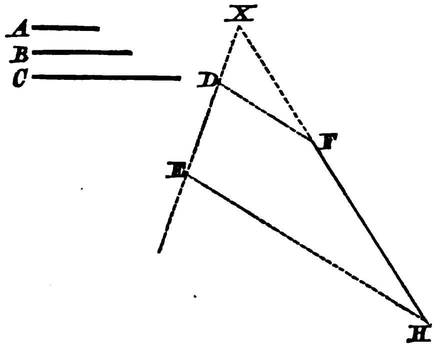
PROBLEM 2.

To find a fourth proportional to three given straight lines.

Let A, B, C be the given straight lines to which it is required to find a fourth proportional.

Construction. Take any angle X, and on one of its arms take XD, DE equal to A, B respectively: and on the other

arm take XF equal to C. Join DF, and draw EH parallel to DF, to meet XF produced in H.



Then shall FH be the line required.

Proof. For since DF is parallel to EH,

$$XD : DE :: XF : FH,$$

but XD, DE, and XF are equal to A, B, C respectively; therefore $$A : B :: C : FH;$$
that is, FH is the fourth proportional required.

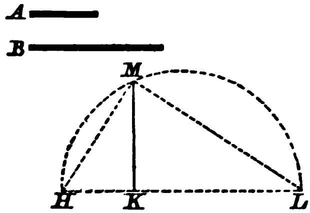
Cor. *Hence a third proportional to two given straight lines can be found, by taking $C = B$.*

Problem 3.

To find a mean proportional between two given straight lines.

Let A, B be the given straight lines: it is required to find a mean proportional between A and B.

Construction. Take HK, KL in the same straight line, equal to A and B respectively. On HK describe a semi-



circle, and draw KM perpendicular to HL to meet the circumference in N. KM is the line required.

Proof. Join HM, ML. Then since HML is a semicircle, HML is a right angle; therefore MK, the perpendicular from the right angle on the hypothenuse, is a mean proportional between the segments of the base; that is, MK is a mean proportional between HK and KL, or between A and B.

Problem 4.

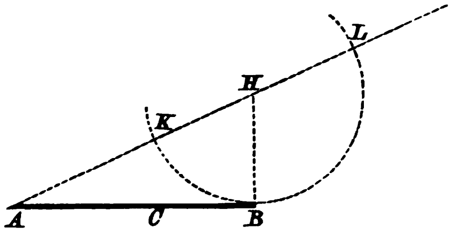
To divide a straight line in extreme and mean ratio.

A line AB is said to be divided in extreme and mean ratio in C, when

$$AB : AC :: AC : CB.$$

Let AB be the given straight line, which it is required to divide in extreme and mean ratio.

Construction. From B draw BH perpendicular to AB, and equal to half AB. Join AH. With centre H, and



radius HB describe a circle cutting AH and AH produced in K and L; and cut off from AB a part $AC = AK$.

C shall be the point required.

Proof. Since HB is at right angles to AB, AB is a tangent to the circle KBL.

Therefore $\quad AL : AB :: AB : AK.$

Therefore

$$AB : AL - AB :: AK : AB - AK \text{ (III. 3, Cor. 3).}$$

But since $AB = 2HB = KL$, and $AK = AC$,

we have $\quad AL - AB = AK = AC,$

and $\quad AB - AK = BC;$

therefore $\quad AB : AC :: AC : BC.$

This problem is solved as an exercise in Book I. Section VI.

Algebraical Solution.

Remark. It is instructive to compare this with the algebraical solution. Let the given line $AB = a$, that is, contain a units of length: and let AC the portion required, which is at present unknown, $= x$.

Then by the conditions $a : x :: x : a-x$,

that is, $$a(a-x) = x^2.$$

But this is a quadratic equation in x, solving which we obtain

$$x = \frac{\pm\sqrt{5}-1}{2} \cdot a.$$

And the geometrical construction is the expression of the operations here represented. For if $AB = a$, $HB = \frac{1}{2}a$, and $AH^2 = AB^2 + HB^2 = a^2 + \frac{1}{4}a^2 = \frac{5}{4}a^2$; and $AH = \frac{\sqrt{5}}{2} \cdot a$; and $HK = \frac{1}{2}a$; and therefore $AK = \frac{\sqrt{5}}{2}a - \frac{1}{2}a = \frac{\sqrt{5}-1}{2}a$.

Therefore $AC = x = \dfrac{\sqrt{5}-1}{2}a$.

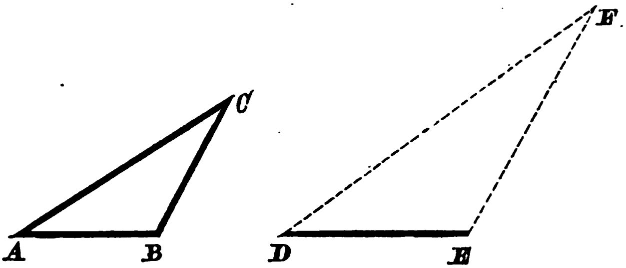
The negative sign before the radical corresponds to the solution of a problem more general than the geometrical problem as stated above, and the consideration of it must be deferred.

PROBLEM 5.

To construct a triangle similar to a given triangle, on a straight line which is to be homologous to a given side of the triangle.

Let ABC be the given triangle, DE the given straight line which is to be homologous to AB.

Construction. At D and E draw lines making with DE angles equal to the angles A and B, and let these lines meet in F. Then DEF is the triangle required.



Proof. For the triangle DEF is by construction equiangular to the triangle ABC, and therefore it is similar to it.

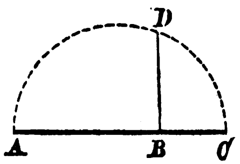
Cor. *Hence a polygon can be constructed similar to any given polygon, on a straight line which is to be homologous to a given side of the polygon.*

For the given polygon can be divided into triangles.

PROBLEM 6.

To make a square which shall be to a given square in a given ratio.

Let AB be the side of the given square, $AB : BC$ in the given ratio.



Construction. On AC describe a semicircle, and draw BD perpendicular to AC, to meet the semicircle in D. BD is the side of the square required.

Proof. Since $AB : BD :: BD : BC$,
therefore $AB^2 : BD^2 :: AB : BC$,
that is $AB^2 : BD^2$ in the given ratio.

COR. *The same construction and proof are applicable to any polygon.*

For similar polygons are to one another as the squares on their homologous sides.

PROBLEM 7.

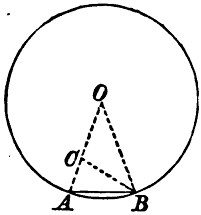
To inscribe a regular decagon in a given circle.

We shall solve this by the method of analysis and synthesis.

Analysis. The radii to two consecutive angular points must include an angle equal to $\frac{1}{10}$th of 4 right angles, or to $\frac{1}{5}$th of 2 right angles. That is, if OA, OB are such radii, the angle at O must be $\frac{1}{5}$th of two right angles, and therefore the angles at A and B must each be $\frac{2}{5}$ths of 2 right angles, since the three angles are together equal to 2 right angles.

Bisect the angle OBA by the line BC. Then OBC and CBA each are equal to AOB; and therefore $CB = CO$,

and BCA is double of COB, and therefore $= BAC$; therefore $AB = BC = CO$.



But since OBA is bisected by BC

$$OB : AB :: OC : CA;$$

therefore $$OA : OC :: OC : CA,$$

or OA is divided in extreme and mean ratio in C.

Hence the construction follows.

Synthesis. Take OA any radius of the circle; divide it in extreme and mean ratio in C so that

$$OA : OC :: OC : CA.$$

Place AB as a chord of the circle equal to OC. Join BO, CA.

Then AB is a side of a decagon inscribed in the circle.

Proof. For since $OA : OC :: OC : CA$,

and $$OA = OB, \text{ and } OC = AB;$$

therefore $$OB : BA :: OC : CA;$$

and therefore CB bisects the angle OBA; and the angle OBA is double of ABC.

And again since the triangles OAB, BAC have the angle at A common, and have the sides about the common angle proportionals, viz. $OA : AB :: AB : AC$;

Therefore these triangles are similar and equiangular; therefore the angle ABC is equal to the angle AOB.

But OBA is double of ABC, and therefore each of the angles OBA, OAB is double of AOB.

But the three angles OBA, OAB, AOB together equal 2 right angles; therefore $OAB = \frac{1}{5}$th of 2 right angles, or $\frac{1}{10}$th of 4 right angles.

Hence ten chords equal to AB could be placed round the circumference of the circle, and thus a regular decagon would be inscribed.

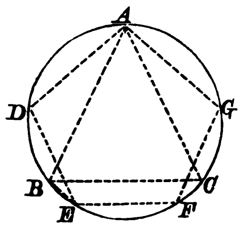
COR. 1. *Hence a regular pentagon may be inscribed in a circle by joining the alternate angular points of an inscribed decagon.*

COR. 2. *Hence regular polygons of* 20, 40, 80 *sides can be constructed.*

PROBLEM 8.

To inscribe a regular quindecagon in a given circle.

Let ABC be the given circle, in which it is required to inscribe a regular quindecagon.



Construction. Inscribe an equilateral triangle ABC, and a regular pentagon $ADEFG$, having one angular point A common.

Join BE, and place chords equal to BE round the circumference of the circle: they will form an inscribed quindecagon.

Proof. For AE is $\frac{2}{5}$ths of the circumference, and AB is $\frac{1}{3}$rd of the circumference; therefore BE is $\frac{2}{5} - \frac{1}{3} = \frac{1}{15}$th of the circumference, and therefore 15 chords equal to BE can be placed round the circumference of the circle, and will form a regular quindecagon.

Cor. *Hence regular polygons of* 30, 60, 120 &c. *sides can be constructed.*

Regular polygons can therefore be constructed when the number of their sides is 3, 4, 5, or 15, or these numbers multiplied by any power of 2. And besides these no other regular polygons can be constructed by the use of the ruler and compasses only, with the remarkable exception discovered by Gauss; who shewed that a polygon of $2^n + 1$ sides can be described by the ruler and compasses alone, when n is such that $2^n + 1$ is a prime number. If n has the values 1, 2, 3 ... in succession, $2^n + 1$ takes the values 3, 5, 9, 17, 33, 65, 129, 257 ... of which 3, 5, 17, 257 are primes. Hence Gauss has shewn that regular polygons of 17 and 257 sides *can* be constructed by the use of the ruler and compasses; but the construction and proof, even for the first of these, are far too tedious to be given in an elementary work.

Miscellaneous Theorems and Problems.

1. The bisector of an angle of an equilateral triangle, passes through one of the points of trisection of the perpendicular from either of the other angles on the opposite side.

2. The bisectors of the angles of a triangle intersect in one point.

3. ABC, PQR are two parallel lines such that

$$AB : BC :: PQ : QR,$$

prove that AP, BQ, CR are either parallel or meet in one point.

4. The external bisector of the vertical angle of an isosceles triangle is parallel to the base.

5. The line joining the middle points of the sides of a triangle is parallel to the base, and is equal to half the base.

6. The triangle formed by joining the middle points of the sides of a triangle is similar to the original triangle, and has one fourth of its area.

7. The lines that join the middle points of adjacent sides of a quadrilateral form a parallelogram. Under what circumstances will it be a rhombus, a square, or a rectangle?

8. CAB is a triangle, and in AC a point A' is taken, and BB' is cut off from CB produced, so that $AA' = BB'$. Prove that $A'B'$ is cut by AB into parts which have to one another the ratio $CB : CA$.

9. To inscribe a square in a triangle.

10. If two triangles are on equal bases between the same parallels any straight line parallel to their bases will cut off equivalent areas from the two triangles.

11. Make an equilateral triangle equivalent to a given square.

12. Find a point O within the triangle ABC, such that OAB, OAC, OBC shall be equivalent triangles.

13. The angle A of a triangle ABC is bisected by a line that meets the base in D: BC is bisected in O. Prove that $OB : OD :: AB + AC : AB \sim AC$.

14. Given the base, vertical angle, and ratio of the sides, construct the triangle.

15. Perpendiculars are drawn from any point within an equilateral triangle on the three sides; shew that their sum is invariable.

16. Deduce from Ptolemy's Theorem that if P is any point in the circumference of the circle circumscribing an equilateral triangle ABC, of the three lines PA, PB, PC one is equal to the sum of the other two.

17. From any point in the base of a triangle lines are drawn parallel to the two sides. Find the locus of the intersection of the diagonals of the parallelograms so formed.

18. Let P, Q be points in AB, and AB produced, so that $AP : PB :: AQ : QB$; through B draw a perpendicular to AB to meet the semicircle on PQ in M: prove that AM touches the circle at M.

19. AB is a given line, and CD a given length on a line parallel to AB, and AC, BD intersect in O; prove that as CD varies in position, the locus of O is a line parallel to AB.

20. AB is a diameter of a circle of which AEF, BEG are chords. CED is drawn through E at right angles to AB: prove that $CFDG$ is a quadrilateral such that the ratio of any pair of its adjacent sides is equal to the ratio of the other pair.

21. Divide a given arc of a circle into two parts which have their chords in a given ratio to one another.

22. If in two similar triangles lines are drawn from two of the equal angles to make equal angles with the homologous sides, these lines shall have to one another the same ratio as the sides of the triangle.

23. To make a rectilineal figure similar to a given rectilineal figure, and having a given ratio to it.

24. To find two straight lines which shall have the same ratio as two given rectangles.

25. To describe on a given straight line a rectangle equal to a given rectangle.

26. To make an isosceles triangle, with a given vertical angle, equal to a given triangle.

27. In a quadrilateral figure which cannot be inscribed in a circle the rectangle contained by the diagonals is less than the sum of the rectangles contained by the opposite sides.

28. In any triangle ABC the rectangle $AB \times AC$ is equal to the rectangle contained by the diameter of the circle circumscribing the triangle, and the perpendicular from A on BC.

29. Hence shew that if A be the area of a triangle ABC, D the diameter of the circumscribing circle,

$$A \times D = \tfrac{1}{2} AB \times BC \times CA.$$

30. Construct a rectangle equal to a given square, and having the sum of its adjacent sides equal to a given straight line.

31. Construct a rectangle equal to a given square, and having the difference of its adjacent sides equal to a given square.

32. Describe a rectangle equal to a given square, and having its sides in a given ratio.

33. To make a figure similar to a given figure, and having a given ratio to it.

34. AB is a diameter of a circle, and at A and B tangents are drawn to the circle. If PCQ be a tangent at any point C, cutting the tangents at A, B in P, Q, prove that the radius of the circle is a mean proportional between the segments PC, QC.

35. With the same figure prove that if AQ, BP intersect in R, then CR is parallel to AP or BQ.

36. If two triangles AEF, ABC have a common angle A, prove that

the triangle AEF : triangle $ABC = AE \,.\, AF : AB \,.\, AC$.

37. Given two points in a terminated straight line, find a point in the straight line such that its distances from the extremities of the line are to one another in the same ratio as its distances from the fixed points.

38. Divide a given straight line into two parts such that their squares may have a given ratio to one another.

39. AB is divided in C; shew that the perpendiculars from A, B on any straight line through C have to one another a constant ratio.

40. From the obtuse angle of a triangle to draw a line to the base which shall be a mean proportional between the segments of the base.

41. Divide a given triangle into two parts which shall have to one another a given ratio by a line parallel to one of the sides.

42. If from any point in the circumference of a circle perpendiculars be drawn to the sides, or sides produced of an inscribed triangle, prove that the feet of these perpendiculars lie in one straight line.

43. If a line be divided into any two parts to find the locus of the point in which these parts subtend equal angles.

44. If two circles touch each other externally, and also touch a straight line, prove that the part of the line between the points of contact is a mean proportional between the diameters of the circles.

45. Any regular polygon inscribed in a circle is a mean proportional between the inscribed and circumscribed regular polygons of half the number of sides.

46. ABC is a triangle, and O is the point of intersection of the perpendicular from ABC on the opposite sides of the triangle: the circle which passes through the middle points of OA, OB, OC, will pass through the feet of the perpendiculars, and through the middle points of the sides of the triangle.

47. Describe a circle to touch a given straight line and a given circle, and to pass through a given point.

48. A and B are two points on the same side of a straight line which meet AB produced in C. Of all the points in this straight line find that at which AB subtends the greatest angle.

49. Inscribe a square in a given pentagon.

50. $ABCD$ is a quadrilateral figure circumscribing a circle, and through the centre O, a line EOF equally inclined to AB and BC is drawn to meet them in E and F: prove that $AE : EB :: CF : FD$.

CAMBRIDGE: PRINTED BY C. J. CLAY, M.A. AT THE UNIVERSITY PRESS.

NOVEMBER, 1868.

LIST OF EDUCATIONAL BOOKS

PUBLISHED BY

MACMILLAN AND CO.,

16, *BEDFORD STREET, COVENT GARDEN,*

London, W.C.

CONTENTS.

MESSRS. MACMILLAN & CO. *beg to call attention to the accompanying Catalogue of their* EDUCATIONAL WORKS, *the writers of which are mostly scholars of eminence in the Universities, as well as of large experience in teaching.*

Many of the works have already attained a wide circulation in England and in the Colonies, and are acknowledged to be among the very best Educational Books on their respective subjects.

The books can generally be procured by ordering them through local booksellers in town or country, but if at any time difficulty should arise, Messrs. MACMILLAN *will feel much obliged by direct communication with themselves on the subject.*

Notices of errors or defects in any of these works will be gratefully received and acknowledged.

LIST OF EDUCATIONAL BOOKS.

CLASSICAL.

ÆSCHYLI EUMENIDES. The Greek Text, with English Notes, and English Verse Translation and an Introduction. By BERNARD DRAKE, M.A., late Fellow of King's College, Cambridge. 8vo. 7*s.* 6*d.*

The Greek Text adopted in this Edition is based upon that of Wellauer, which may be said in general terms to represent that of the best manuscripts. But in correcting the Text, and in the Notes, advantage has been taken of the suggestions of Hermann, Paley, Linwood, and other commentators.

ARISTOTLE ON FALLACIES; OR, THE SOPHISTICI ELENCHI. With a Translation and Notes by EDWARD POSTE, M.A., Fellow of Oriel College, Oxford. 8vo. 8*s.* 6*d.*

Besides the doctrine of Fallacies, Aristotle offers either in this treatise, or in other passages quoted in the commentary, various glances over the world of science and opinion, various suggestions on problems which are still agitated, and a vivid picture of the ancient system of dialectics, which it is hoped may be found both interesting and instructive.

"It is not only scholarlike and careful; it is also perspicuous."—*Guardian.*

ARISTOTLE.—AN INTRODUCTION TO ARISTOTLE'S RHETORIC. With Analysis, Notes, and Appendices. By E. M. COPE, Senior Fellow and Tutor of Trinity College, Cambridge. 8vo. 14*s.*

This work is introductory to an edition of the Greek Text of Aristotle's Rhetoric, which is in course of preparation.

"Mr. Cope has given a very useful appendage to the promised Greek Text; but also a work of so much independent use that he is quite justified in his separate publication. All who have the Greek Text will find themselves supplied with a comment; and those who have not will find an analysis of the work."—*Athenæum.*

CATULLI VERONENSIS LIBER, edited by R. ELLIS, Fellow of Trinity College, Oxford. 18mo. 3*s*. 6*d*.

"It is little to say that no edition of Catullus at once so scholarlike has ever appeared in England."—*Athenæum.*

"Rarely have we read a classic author with so reliable, acute, and safe a guide."—*Saturday Review.*

CICERO.—THE SECOND PHILIPPIC ORATION. With an Introduction and Notes, translated from the German of KARL HALM. Edited, with Corrections and Additions, by JOHN E. B. MAYOR, M.A., Fellow and Classical Lecturer of St. John's College, Cambridge. Third Edition, revised. Fcap. 8vo. 5*s*.

"A very valuable edition, from which the student may gather much both in the way of information directly communicated, and directions to other sources of knowledge."—*Athenæum.*

DEMOSTHENES ON THE CROWN. The Greek Text with English Notes. By B. DRAKE, M.A., late Fellow of King's College, Cambridge. Third Edition, to which is prefixed ÆSCHINES AGAINST CTESIPHON, with English Notes. Fcap. 8vo. 5*s*.

The terseness and felicity of Mr. Drake's translations constitute perhaps the chief value of his edition, and the historical and archæological details necessary to understanding the *De Coronâ* have in some measure been anticipated in the notes on the Oration of Æschines. In both, the text adopted in the Zurich edition of 1851, and taken from the Parisian MS., has been adhered to without any variation. Where the readings of Bekker, Dissen, and others appear preferable, they are subjoined in the notes.

HODGSON.—MYTHOLOGY FOR LATIN VERSIFICATION. A Brief Sketch of the Fables of the Ancients, prepared to be rendered into Latin Verse for Schools. By F. HODGSON, B.D., late Provost of Eton. New Edition, revised by F. C. HODGSON, M.A. 18mo. 3*s*.

Intending the little book to be entirely elementary, the Author has made it as easy as he could, without too largely superseding the use of the Dictionary and Gradus. By the facilities here afforded, it will be possible, in many cases, for a boy to get rapidly through these preparatory exercises; and thus, having mastered the first difficulties, he may advance with better hopes of improvement to subjects of higher character, and verses of more difficult composition.

JUVENAL, FOR SCHOOLS. With English Notes. By J. E. B. MAYOR, M.A. New and Cheaper Edition. Crown 8vo. [*In the Press.*

"A School edition of Juvenal, which, for really ripe scholarship, extensive acquaintance with Latin literature, and familiar knowledge of Continental criticism, ancient and modern, is unsurpassed, we do not say among English School-books, but among English editions generally."—*Edinburgh Review.*

LYTTELTON.—THE COMUS of MILTON rendered into Greek Verse. By LORD LYTTELTON. Extra fcap. 8vo. Second Edition. 5*s.*

— THE SAMSON AGONISTES of MILTON rendered into Greek Verse. By LORD LYTTELTON. Extra fcap. 8vo. 6*s.* 6*d.*

MARSHALL.—A TABLE OF IRREGULAR GREEK VERBS, Classified according to the Arrangement of Curtius's Greek Grammar. By J. M. MARSHALL, M.A., Fellow and late Lecturer of Brasenose College, Oxford; one of the Masters in Clifton College. 8vo. cloth. 1*s.*

MAYOR.—FIRST GREEK READER. Edited after KARL HALM, with Corrections and large Additions by JOHN E. B. MAYOR, M.A., Fellow and Classical Lecturer of St. John's College, Cambridge. Fcap. 8vo. 6*s.*

MERIVALE.—KEATS' HYPERION rendered into Latin Verse. By C. MERIVALE, B.D. Second Edition. Extra fcap. 8vo. 3*s.* 6*d.*

PLATO.—THE REPUBLIC OF PLATO. Translated into English, with an Analysis and Notes, by J. Ll. DAVIES, M.A., and D. J. VAUGHAN, M.A. Third Edition, with Vignette Portraits of Plato and Socrates, engraved by JEENS from an Antique Gem. 18mo. 4*s.* 6*d.*

ROBY.—A LATIN GRAMMAR for the Higher Classes in Grammar Schools. By H. J. ROBY, M.A.; based on the "Elementary Latin Grammar." [*In the Press.*

SALLUST.—CAII SALLUSTII CRISPI Catilina et Jugurtha. For use in Schools (with copious Notes). By C. MERIVALE, B.D. (In the present Edition the Notes have been carefully revised, and a few remarks and explanations added.) Second Edition. Fcap. 8vo. 4*s*. 6*d*.

The Jugurtha and the Catilina may be had separately, price 2*s*. 6*d*. each.

TACITUS.—THE HISTORY OF TACITUS translated into ENGLISH. By A. J. CHURCH, M.A., and W. J. BRODRIBB, M.A. With Notes and a Map. 8vo. 10*s*. 6*d*.

The translators have endeavoured to adhere as closely to the original as was thought consistent with a proper observance of English idiom. At the same time it has been their aim to reproduce the precise expressions of the author. The campaign of Civilis is elucidated in a note of some length which is illustrated by a map, containing only the names of places and of tribes occurring in the work.

— THE AGRICOLA and GERMANY. By the same translators. With Maps and Notes. Extra fcap. 8vo. 2*s*. 6*d*.

THRING.—Works by **Edward Thring, M.A.**, Head Master of Uppingham School:—

— A CONSTRUING BOOK. Fcap. 8vo. 2*s*. 6*d*.

This Construing Book is drawn up on the same sort of graduated scale as the Author's *English Grammar*. Passages out of the best Latin Poets are gradually built up into their perfect shape. The few words altered, or inserted as the passages go on, are printed in Italics. It is hoped by this plan that the learner, whilst acquiring the rudiments of language, may store his mind with good poetry and a good vocabulary.

— A LATIN GRADUAL. A First Latin Construing Book for Beginners. Fcap. 8vo. 2*s*. 6*d*.

The main plan of this little work has been well tested.

The intention is to supply by easy steps a knowledge of Grammar, combined with a good vocabulary; in a word, a book which will not require to be forgotten again as the learner advances.

A short practical manual of common Mood constructions, with their English equivalents, form the second part.

— A MANUAL of MOOD CONSTRUCTIONS. Extra fcap. 8vo. 1*s*. 6*d*.

THUCYDIDES.—THE SICILIAN EXPEDITION. Being Books VI. and VII. of Thucydides, with Notes. A New Edition, revised and enlarged, with a Map. By the Rev. PERCIVAL FROST, M.A., late Fellow of St. John's College, Cambridge. Fcap. 8vo. 5*s*.

This edition is mainly a grammatical one. Attention is called to the force of compound verbs, and the exact meaning of the various tenses employed.

WRIGHT.—Works by **J. Wright, M.A.,** late Head Master of Sutton Coldfield School :—

— HELLENICA; Or, a HISTORY of GREECE in GREEK, as related by Diodorus and Thucydides, being a First Greek Reading Book, with Explanatory Notes Critical and Historical. Second Edition, with a Vocabulary. 12mo. 3*s.* 6*d.*

In the last twenty chapters of this volume, Thucydides sketches the rise and progress of the Athenian Empire in so clear a style and in such simple language, that the author doubts whether any easier or more instructive passages can be selected for the use of the pupil who is commencing Greek.

— A HELP TO LATIN GRAMMAR; Or, the Form and Use of Words in Latin, with Progressive Exercises. Crown 8vo. 4*s.* 6*d.*

"Never was there a better aid offered alike to teacher and scholar in that arduous pass. The style is at once familiar and strikingly simple and lucid; and the explanations precisely hit the difficulties, and thoroughly explain them."—*English Journal of Education.*

— THE SEVEN KINGS OF ROME. An Easy Narrative, abridged from the First Book of Livy by the omission of difficult passages, being a First Latin Reading Book, with Grammatical Notes. Fcap. 8vo. 3*s.*

This work is intended to supply the pupil with an easy Construing-book, which may at the same time be made the vehicle for instructing him in the rules of grammar and principles of composition. Here Livy tells his own pleasant stories in his own pleasant words. Let Livy be the master to teach a boy Latin, not some English collector of sentences, and he will not be found a dull one.

— A VOCABULARY AND EXERCISES on the "SEVEN KINGS OF ROME." Fcap. 8vo. 2*s.* 6*d.*

The Vocabulary and Exercises may also be had bound up with "The Seven Kings of Rome," price 5*s.*

MATHEMATICAL.

AIRY.—Works by **G. B. Airy,** Astronomer Royal :—

— ELEMENTARY TREATISE ON PARTIAL DIFFERENTIAL EQUATIONS. Designed for the use of Students in the University. With Diagrams. Crown 8vo. cloth, 5*s.* 6*d.*

It is hoped that the methods of solution here explained, and the instances exhibited, will be found sufficient for application to nearly all the important problems of Physical Science, which require for their complete investigation the aid of partial differential equations.

AIRY.—Works by **G. B. Airy**—*Continued.*

— ON THE ALGEBRAICAL AND NUMERICAL THEORY of ERRORS of OBSERVATIONS, and the COMBINATION of OBSERVATIONS. Crown 8vo. cloth, 6*s.* 6*d.*

— UNDULATORY THEORY OF OPTICS. Designed for the use of Students in the University. New Edition. Crown 8vo. cloth, 6*s.* 6*d.*

— ON SOUND and ATMOSPHERIC VIBRATIONS. With the Mathematical Elements of Music. Designed for the use of Students of the University. Crown 8vo. 9*s.*

BAYMA.—THE ELEMENTS of MOLECULAR MECHANICS. By JOSEPH BAYMA, S.J., Professor of Philosophy, Stonyhurst College. Demy 8vo. cloth, 10*s.* 6*d.*

BOOLE.—Works by **G. Boole, D.C.L., F.R.S.,** Professor of Mathematics in the Queen's University, Ireland:—

— A TREATISE ON DIFFERENTIAL EQUATIONS. New and Revised Edition. Edited by I. TODHUNTER. Crown 8vo. cloth, 14*s.*

The author has endeavoured in this Treatise to convey as complete an account of the present state of knowledge on the subject of Differential Equations, as was consistent with the idea of a work intended primarily for elementary instruction. The earlier sections of each chapter contain that kind of matter which has usually been thought suitable to the beginner, while the later ones are devoted either to an account of recent discovery, or the discussion of such deeper questions of principle as are likely to present themselves to the reflective student in connexion with the methods and processes of his previous course.

— A TREATISE ON DIFFERENTIAL EQUATIONS. Supplementary Volume. Edited by I. TODHUNTER. Crown 8vo. cloth, 8*s.* 6*d.*

— THE CALCULUS OF FINITE DIFFERENCES. Crown 8vo. cloth, 10*s.* 6*d.*

This work is in some measure designed as a sequel to the *Treatise on Differential Equations,* and is composed on the same plan.

BEASLEY.—AN ELEMENTARY TREATISE ON PLANE TRIGONOMETRY. With Examples. By R. D. BEASLEY, M.A., Head Master of Grantham Grammar School. Second Edition, revised and enlarged. Crown 8vo. cloth, 3*s*. 6*d*.

This Treatise is specially intended for use in Schools. The choice of matter has been chiefly guided by the requirements of the three days' Examination at Cambridge, with the exception of proportional parts in Logarithms, which have been omitted. About *Four hundred* Examples have been added, mainly collected from the Examination Papers of the last ten years, and great pains have been taken to exclude from the body of the work any which might dishearten a beginner by their difficulty.

CAMBRIDGE SENATE-HOUSE PROBLEMS and RIDERS, WITH SOLUTIONS:—

1848—1851.—PROBLEMS. By FERRERS and JACKSON. 8vo. cloth. 15*s*. 6*d*.

1848—1851.—RIDERS. By JAMESON. 8vo. cloth. 7*s*. 6*d*.

1854.—PROBLEMS and RIDERS. By WALTON and MACKENZIE, 8vo. cloth. 10*s*. 6*d*.

1857.—PROBLEMS and RIDERS. By CAMPION and WALTON. 8vo. cloth. 8*s*. 6*d*.

1860.—PROBLEMS and RIDERS. By WATSON and ROUTH. Crown 8vo. cloth. 7*s*. 6*d*.

1864.—PROBLEMS and RIDERS. By WALTON and WILKINSON. 8vo. cloth. 10*s*. 6*d*.

CAMBRIDGE COURSE OF ELEMENTARY NATURAL PHILOSOPHY, for the Degree of B.A. Originally compiled by J. C. SNOWBALL, M.A., late Fellow of St. John's College. Fifth Edition, revised and enlarged, and adapted for the Middle-Class Examinations by THOMAS LUND, B.D., Late Fellow and Lecturer of St. John's College; Editor of Wood's Algebra, &c. Crown 8vo. cloth. 5*s*.

This work will be found suited to the wants, not only of University Students, but also of many others who require a short course of Mechanics and Hydrostatics, and especially of the Candidates at our Middle-Class Examinations.

CAMBRIDGE AND DUBLIN MATHEMATICAL JOURNAL. THE COMPLETE WORK, in Nine Vols. 8vo. cloth. £7 4*s*.
(Only a few copies remain on hand.)

CHEYNE.—AN ELEMENTARY TREATISE on the PLANETARY THEORY. With a Collection of Problems. By C. H. H. CHEYNE, B.A. Crown 8vo. cloth. 6*s*. 6*d*.

— THE EARTH'S MOTION of ROTATION. By C. H. H. CHEYNE, M.A. Crown 8vo. 3*s*. 6*d*.

CHILDE.—THE SINGULAR PROPERTIES of the ELLIPSOID and ASSOCIATED SURFACES of the Nth DEGREE. By the Rev. G. F. CHILDE, M.A., Author of "Ray Surfaces," "Related Caustics," &c. 8vo. 10*s.* 6*d.*

CHRISTIE.—A COLLECTION OF ELEMENTARY TEST-QUESTIONS in PURE and MIXED MATHEMATICS; with Answers and Appendices on Synthetic Division, and on the Solution of Numerical Equations by Horner's Method. By JAMES R. CHRISTIE, F.R.S., late First Mathematical Master at the Royal Military Academy, Woolwich. Crown 8vo. cloth, 8*s.* 6*d.*

DALTON.—ARITHMETICAL EXAMPLES. Progressively arranged, with Exercises and Examination Papers. By the Rev. T. DALTON, M.A., Assistant Master of Eton College. 18mo. cloth. 2*s.* 6*d.*

DAY.—PROPERTIES OF CONIC SECTIONS PROVED GEOMETRICALLY. Part I., THE ELLIPSE, with Problems. By the Rev. H. G. DAY, M.A., Head Master of Sedbergh Grammar School. Crown 8vo. 3*s.* 6*d.*

DODGSON.—AN ELEMENTARY TREATISE ON DETERMINANTS, with their Application to Simultaneous Linear Equations and Algebraical Geometry. By C. L. DODGSON, M.A., Mathematical Lecturer of Christ Church, Oxford. Small 4to. cloth, 10*s.* 6*d.*

DREW.—GEOMETRICAL TREATISE on CONIC SECTIONS. By W. H. DREW, M.A., St. John's College, Cambridge. Third Edition. Crown 8vo. cloth, 4*s.* 6*d.*

In this work the subject of Conic Sections has been placed before the student in such a form that, it is hoped, after mastering the elements of Euclid, he may find it an easy and interesting continuation of his geometrical studies. With a view also of rendering the work a complete Manual of what is required at the Universities, there have been either embodied into the text, or inserted among the examples, every book-work question, problem, and rider, which has been proposed in the Cambridge examinations up to the present time.

— SOLUTIONS TO THE PROBLEMS IN DREW'S CONIC SECTIONS. Crown 8vo. cloth, 4*s.* 6*d.*

FERRERS.—AN ELEMENTARY TREATISE on TRILINEAR CO-ORDINATES, the Method of Reciprocal Polars, and the Theory of Projections. By the Rev. N. M. FERRERS, M.A., Fellow and Tutor of Gonville and Caius College, Cambridge. Second Edition. Crown 8vo. 6*s.* 6*d.*

The object of the author in writing on this subject has mainly been to place it on a basis altogether independent of the ordinary Cartesian system, instead of regarding it as only a special form of Abridged Notation. A short chapter on Determinants has been introduced.

FROST.—THE FIRST THREE SECTIONS of NEWTON'S PRINCIPIA. With Notes and Illustrations. Also a Collection of Problems, principally intended as Examples of Newton's Methods. By PERCIVAL FROST, M.A., late Fellow of St. John's College, Mathematical Lecturer of King's College, Cambridge. Second Edition. 8vo. cloth, 10*s.* 6*d.*

The author's principal intention is to explain difficulties which may be encountered by the student on first reading the Principia, and to illustrate the advantages of a careful study of the methods employed by Newton, by showing the extent to which they may be applied in the solution of problems; he has also endeavoured to give assistance to the student who is engaged in the study of the higher branches of Mathematics, by representing in a geometrical form several of the processes employed in the Differential and Integral Calculus, and in the analytical investigations of Dynamics.

FROST and WOLSTENHOLME.—A TREATISE ON SOLID GEOMETRY. By PERCIVAL FROST, M.A., and the Rev. J. WOLSTENHOLME, M.A., Fellow and Assistant Tutor of Christ's College. 8vo. cloth, 18*s.*

The authors have endeavoured to present before students as comprehensive a view of the subject as possible. Intending as they have done to make the subject accessible, at least in the earlier portion, to all classes of students, they have endeavoured to explain fully all the processes which are most useful in dealing with ordinary theorems and problems, thus directing the student to the selection of methods which are best adapted to the exigencies of each problem. In the more difficult portions of the subject, they have considered themselves to be addressing a higher class of students; there they have tried to lay a good foundation on which to build, if any reader should wish to pursue the science beyond the limits to which the work extends.

GODFRAY.—A TREATISE on ASTRONOMY, for the use of Colleges and Schools. By HUGH GODFRAY, M.A., Mathematical Lecturer at Pembroke College, Cambridge. 8vo. cloth. 12*s.* 6*d.*

"We can recommend for its purpose a very good *Treatise on Astronomy* by Mr. Godfray. It is a working book, taking astronomy in its proper place in mathematical science. But it begins with the elementary definitions, and connects the mathematical formulæ very clearly with the visible aspect of the heavens and the instruments which are used for observing it."—*Guardian.*

— AN ELEMENTARY TREATISE on the LUNAR THEORY. With a brief Sketch of the Problem up to the time of Newton. By HUGH GODFRAY, M.A. Second Edition, revised. Crown 8vo. cloth. 5*s.* 6*d.*

HEMMING.—AN ELEMENTARY TREATISE on the DIFFERENTIAL AND INTEGRAL CALCULUS, for the use of Colleges and Schools. By G. W. HEMMING, M.A., Fellow of St. John's College, Cambridge. Second Edition, with Corrections and Additions. 8vo. cloth. 9*s.*

JONES and CHEYNE.—ALGEBRAICAL EXERCISES. Progressively arranged. By the Rev. C. A. JONES, M.A., and C. H. CHEYNE, M.A., Mathematical Masters of Westminster School. New Edition. 18mo. cloth, 2*s.* 6*d.*

This little book is intended to meet a difficulty which is probably felt more or less by all engaged in teaching Algebra to beginners. It is that while new ideas are being acquired, old ones are forgotten. In the belief that constant practice is the only remedy for this, the present series of miscellaneous exercises has been prepared. Their peculiarity consists in this, that though miscellaneous they are yet progressive, and may be used by the pupil almost from the commencement of his studies. They are not intended to supersede the systematically arranged examples to be found in ordinary treatises on Algebra, but rather to supplement them.

The book being intended chiefly for Schools and Junior Students, the higher parts of Algebra have not been included.

KITCHENER.—A GEOMETRICAL NOTE-BOOK, containing Easy Problems in Geometrical Drawing preparatory to the Study of Geometry. For the use of Schools. By F. E. KITCHENER, M.A., Mathematical Master at Rugby. 4to. 2*s.*

MORGAN.—A COLLECTION of PROBLEMS and EXAMPLES in Mathematics. With Answers. By H. A. MORGAN, M.A., Sadlerian and Mathematical Lecturer of Jesus College, Cambridge. Crown 8vo. cloth. 6*s.* 6*d.*

This book contains a number of problems, chiefly elementary, in the Mathematical subjects usually read at Cambridge. They have been selected from the papers set during late years at Jesus college. Very few of them are to be met with in other collections, and by far the larger number are due to some of the most distinguished Mathematicians in the University.

PARKINSON.—Works by **S. Parkinson, B.D.,** Fellow and Prælector of St. John's College, Cambridge :—

— AN ELEMENTARY TREATISE ON MECHANICS. For the use of the Junior Classes at the University and the Higher Classes in Schools. With a Collection of Examples. Third Edition, revised. Crown 8vo. cloth, 9*s.* 6*d.*

The author has endeavoured to render the present volume suitable as a Manual for the junior classes in Universities and the higher classes in Schools. In the Third Edition several additional propositions have been incorporated in the work for the purpose of rendering it more complete, and the Collection of Examples and Problems has been largely increased.

— A TREATISE on OPTICS. Second Edition, revised. Crown 8vo. cloth, 10*s.* 6*d.*

A collection of Examples and Problems has been appended to this work which are sufficiently numerous and varied in character to afford useful exercise for the student: for the greater part of them recourse has been had to the Examination Papers set in the University and the several Colleges during the last twenty years.

PHEAR.—ELEMENTARY HYDROSTATICS. With numerous Examples. By J. B. PHEAR, M.A., Fellow and late Assistant Tutor of Clare College, Cambridge. Fourth Edition. Crown 8vo. cloth, 5*s.* 6*d.*

"An excellent Introductory Book. The definitions are very clear; the descriptions and explanations are sufficiently full and intelligible; the investigations are simple and scientific. The examples greatly enhance its value."—*English Journal of Education.*

PRATT.—A TREATISE on ATTRACTIONS, LAPLACE'S FUNCTIONS, and the FIGURE of the EARTH. By JOHN H. PRATT, M.A., Archdeacon of Calcutta, Author of "The Mathematical Principles of Mechanical Philosophy." Third Edition. Crown 8vo. cloth, 6*s.* 6*d.*

PUCKLE.—AN ELEMENTARY TREATISE on CONIC SECTIONS and ALGEBRAIC GEOMETRY. With numerous Examples and hints for their Solution; especially designed for the use of Beginners. By G. H. PUCKLE, M.A., St. John's College, Cambridge, Head Master of Windermere College. Third Edition, enlarged and improved. Crown 8vo. cloth, 7*s.* 6*d.*

The work has been completely re-written, and a considerable amount of new matter has been added, to suit the requirements of the present time.

RAWLINSON.—ELEMENTARY STATICS. By G. RAWLINSON, M.A. Edited by EDWARD STURGES, M.A., of Emmanuel College, Cambridge, and late Professor of the Applied Sciences, Elphinstone College, Bombay. Crown 8vo. cloth. 4*s.* 6*d.*

Published under the authority of H. M. Secretary of State for use in the Government Schools and Colleges in India.

"This Manual may take its place among the most exhaustive, yet clear and simple, we have met with, upon the composition and resolution of forces, equilibrium, and the mechanical powers."—*Oriental Budget.*

REYNOLDS.—MODERN METHODS IN ELEMENTARY GEOMETRY. By E. M. REYNOLDS, M.A., Mathematical Master in Clifton College. Crown 8vo. 3*s.* 6*d.*

ROUTH.—AN ELEMENTARY TREATISE on the DYNAMICS of a SYSTEM of RIGID BODIES. With Examples. By EDWARD JOHN ROUTH, M.A., Fellow and Assistant Tutor of St. Peter's College, Cambridge; Examiner in the University of London. Crown 8vo. cloth, 10*s.* 6*d.*

SMITH.—A TREATISE on ELEMENTARY STATICS. By J. H. SMITH, M.A., Gonville and Caius College, Cambridge. 8vo. 5*s.* 6*d.*

SMITH.—Works by **Barnard Smith, M.A.,** Rector of Glaston, Rutlandshire, late Fellow and Senior Bursar of St. Peter's College, Cambridge:—

— ARITHMETIC and ALGEBRA, in their Principles and Application, with numerous Systematically arranged Examples, taken from the Cambridge Examination Papers, with especial reference to the Ordinary Examination for B.A. Degree. Tenth Edition. Crown 8vo. cloth, 10*s.* 6*d.*

This work is now extensively used in *Schools* and *Colleges* both *at home* and in the *Colonies*. It has also been found of great service for students preparing for the MIDDLE-CLASS AND CIVIL AND MILITARY SERVICE EXAMINATIONS, from the care that has been taken to elucidate the *principles* of all the Rules.

— ARITHMETIC FOR SCHOOLS. New Edition. Crown 8vo. cloth, 4*s.* 6*d.*

— COMPANION to ARITHMETIC for SCHOOLS. [*Preparing.*

— A KEY to the ARITHMETIC for SCHOOLS. Seventh Edition. Crown 8vo., cloth, 8*s.* 6*d.*

— EXERCISES in ARITHMETIC. With Answers. Crown 8vo. limp cloth, 2*s.* 6*d.* Or sold separately, as follows:—Part I. 1*s.*; Part II. 1*s.* ANSWERS, 6*d.*

These Exercises have been published in order to give the pupil examples in every rule of Arithmetic. The greater number have been carefully compiled from the latest University and School Examination Papers.

— SCHOOL CLASS-BOOK of ARITHMETIC. 18mo. cloth, 3*s.* Or sold separately, Parts I. and II. 10*d.* each; Part III. 1*s.*

— KEYS to SCHOOL CLASS-BOOK of ARITHMETIC. Complete in one Volume, 18mo., cloth, 6*s.* 6*d.*; or Parts I., II., and III. 2*s.* 6*d.* each.

— SHILLING BOOK of ARITHMETIC for NATIONAL and ELEMENTARY SCHOOLS. 18mo. cloth. Or separately, Part I. 2*d.*; Part II. 3*d.*; Part III. 7*d.* ANSWERS, 6*d.*

THE SAME, with Answers complete. 18mo. cloth, 1*s.* 6*d.*

— KEY to SHILLING BOOK of ARITHMETIC. 18mo. cloth, 4*s.* 6*d.*

— EXAMINATION PAPERS in ARITHMETIC. In Four Parts. 18mo. cloth, 1*s.* 6*d.* THE SAME, with Answers, 18mo. 1*s.* 9*d.*

— KEY to EXAMINATION PAPERS in ARITHMETIC. 18mo. cloth, 4*s.* 6*d.*

SNOWBALL.—PLANE and SPHERICAL TRIGONOMETRY. With the Construction and Use of Tables of Logarithms. By J. C. SNOWBALL. Tenth Edition. Crown 8vo. cloth, 7*s.* 6*d.*

TAIT and STEELE.—DYNAMICS of a PARTICLE. With Examples. By Professor TAIT and Mr. STEELE. New Edition. Crown 8vo. cloth, 10*s.* 6*d.*

In this Treatise will be found all the ordinary propositions connected with the Dynamics of Particles which can be conveniently deduced without the use of D'Alembert's Principles. Throughout the book will be found a number of illustrative Examples introduced in the text, and for the most part completely worked out; others, with occasional solutions or hints to assist the student, are appended to each Chapter.

TAYLOR.—GEOMETRICAL CONICS; including Anharmonic Ratio and Projection, with numerous Examples. By C. TAYLOR, B.A., Scholar of St. John's College, Cambridge. Crown 8vo. cloth, 7*s.* 6*d.*

TEBAY.—ELEMENTARY MENSURATION for SCHOOLS. With numerous Examples. By SEPTIMUS TEBAY, B.A., Head Master of Queen Elizabeth's Grammar School, Rivington. Extra fcap. 8vo. 3*s.* 6*d.*

TODHUNTER.—Works by **I. Todhunter, M.A., F.R.S.,** Fellow and Principal Mathematical Lecturer of St. John's College, Cambridge:—

— THE ELEMENTS of EUCLID for the use of COLLEGES and SCHOOLS. New Edition. 18mo. cloth, 3*s.* 6*d.*

— ALGEBRA for BEGINNERS. With numerous Examples. New Edition. 18mo. cloth, 2*s.* 6*d.*

— KEY to ALGEBRA for BEGINNERS. Crown 8vo., cl., 6*s.* 6*d.*

— TRIGONOMETRY for BEGINNERS. With numerous Examples. New Edition. 18mo. cloth, 2*s.* 6*d.*

Intended to serve as an introduction to the larger treatise on *Plane Trigonometry*, published by the author. The same plan has been adopted as in the *Algebra for Beginners:* the subject is discussed in short chapters, and a collection of examples is attached to each chapter.

— MECHANICS for BEGINNERS. With numerous Examples. 18mo. cloth, 4*s.* 6*d.*

Intended as a companion to the two preceding books. The work forms an elementary treatise on *Demonstrative* Mechanics. It may be true that this part of mixed mathematics has been sometimes made too abstract and speculative; but it can hardly be doubted that a knowledge of the elements at least of the theory of the subject is extremely valuable even for those who are mainly concerned with practical results. The author has accordingly endeavoured to provide a suitable introduction to the study of applied as well as of theoretical Mechanics.

TODHUNTER.—Works by **I. Todhunter, M.A.**—*Continued.*

— A TREATISE on the DIFFERENTIAL CALCULUS. With Examples. Fourth Edition. Crown 8vo. cloth, 10*s.* 6*d.*

— A TREATISE on the INTEGRAL CALCULUS. Third Edition, revised and enlarged. With Examples. Crown 8vo. cloth, 10*s.* 6*d.*

— A TREATISE on ANALYTICAL STATICS. With Examples. Third Edition, revised and enlarged. Crown 8vo. cloth, 10*s.* 6*d.*

— PLANE CO-ORDINATE GEOMETRY, as applied to the Straight Line and the CONIC SECTIONS. With numerous Examples. Fourth Edition. Crown 8vo. cloth, 7*s.* 6*d.*

— ALGEBRA. For the use of Colleges and Schools. Fourth Edition. Crown 8vo. cloth, 7*s.* 6*d.*

This work contains all the propositions which are usually included in elementary treatises on Algebra, and a large number of *Examples for Exercise.* The author has sought to render the work easily intelligible to students without impairing the accuracy of the demonstrations, or contracting the limits of the subject. The Examples have been selected with a view to illustrate every part of the subject, and as the number of them is about *Sixteen hundred and fifty*, it is hoped they will supply ample exercise for the student. Each set of Examples has been carefully arranged, commencing with very simple exercises, and proceeding gradually to those which are less obvious.

— PLANE TRIGONOMETRY. For Schools and Colleges. Third Edition. Crown 8vo. cloth, 5*s.*

The design of this work has been to render the subject intelligible to beginners, and at the same time to afford the student the opportunity of obtaining all the information which he will require on this branch of Mathematics. Each chapter is followed by a set of Examples; those which are entitled *Miscellaneous Examples*, together with a few in some of the other sets, may be advantageously reserved by the student for exercise after he has made some progress in the subject. In the Second Edition the hints for the solution of the Examples have been considerably increased.

— A TREATISE ON SPHERICAL TRIGONOMETRY. Second Edition, enlarged. Crown 8vo. cloth, 4*s.* 6*d.*

This work is constructed on the same plan as the *Treatise on Plane Trigonometry*, to which it is intended as a sequel. Considerable labour has been expended on the text in order to render it comprehensive and accurate, and the Examples, which have been chiefly selected from University and College Papers, have all been carefully verified.

— EXAMPLES of ANALYTICAL GEOMETRY of THREE DIMENSIONS. Second Edition, revised. Crown 8vo. cloth, 4*s.*

— AN ELEMENTARY TREATISE on the THEORY of EQUATIONS. Second Edition, revised. Crown 8vo. cloth, 7*s.* 6*d.*

WILSON.—ELEMENTARY GEOMETRY. PART I. Angles, Triangles, Parallels, and Equivalent Figures, with the Application to Problems. By J. M. WILSON, M.A., Fellow of St. John's College, Cambridge, and Mathematical Master in Rugby School. Extra fcap. 8vo. 2*s*. 6*d*.

— A TREATISE on DYNAMICS. By W. P. WILSON, M.A., Fellow of St. John's College, Cambridge; and Professor of Mathematics in Queen's College, Belfast. 8vo. 9*s*. 6*d*.

WOLSTENHOLME.—A BOOK of MATHEMATICAL PROBLEMS on subjects included in the Cambridge Course. By JOSEPH WOLSTENHOLME, Fellow of Christ's College, sometime Fellow of St. John's College, and lately Lecturer in Mathematics at Christ's College. Crown 8vo. cloth. 8*s*. 6*d*.

CONTENTS: Geometry (Euclid).—Algebra.—Plane Trigonometry.—Conic Sections, Geometrical.—Conic Sections, Analytical.—Theory of Equations. —Differential Calculus.—Integral Calculus.—Solid Geometry.—Statics.— Dynamics, Elementary.—Newton.—Dynamics of a Point.—Dynamics of a Rigid Body.—Hydrostatics.—Geometrical Optics.—Spherical Trigonometry and Plane Astronomy.

In each subject the order of the Text-Books in general use in the University of Cambridge has been followed, and to some extent the questions have been arranged in order of difficulty. The collection will be found to be unusually copious in problems in the earlier subjects, by which it is designed to make the work useful to mathematical students, not only in the Universities, but in the higher classes of public schools.

SCIENCE.

AIRY.—POPULAR ASTRONOMY. With Illustrations. By G. B. AIRY, Astronomer Royal. Sixth and Cheaper Edition. 18mo. cloth, 4*s*. 6*d*.

"Popular Astronomy in general has many manuals; but none of them supersede the Six Lectures of the Astronomer Royal under that title. Its speciality is the direct way in which every step is referred to the observatory, and in which the methods and instruments by which every observation is made are fully described. This gives a sense of solidity and substance to astronomical statements which is obtainable in no other way."—*Guardian.*

GEIKIE.—ELEMENTARY LESSONS in PHYSICAL GEOLOGY. By ARCHIBALD GEIKIE, F.R.S., Director of the Geological Survey of Scotland. [*Preparing.*

HUXLEY.—LESSONS in ELEMENTARY PHYSIOLOGY. With numerous Illustrations. By T. H. HUXLEY, F.R.S., Professor of Natural History in the Royal School of Mines. Second Edition. 18mo. cloth, 4*s.* 6*d.*

"It is a very small book, but pure gold throughout. There is not a waste sentence, or a superfluous word, and yet it is all clear as daylight. It exacts close attention from the reader, but the attention will be repaid by a real acquisition of knowledge. And though the book is so small, it manages to touch on some of the very highest problems. . . . . The whole book shows how true it is that the most elementary instruction is best given by the highest masters in any science."—*Guardian.*

"The very best descriptions and explanations of the principles of human physiology which have yet been written by an Englishman."—*Saturday Review.*

LOCKYER.—ELEMENTARY LESSONS in ASTRONOMY. With Coloured Diagram of the Spectra of the Sun, Stars, and Nebulæ, and numerous Illustrations. By J. NORMAN LOCKYER, F.R.A.S. 18mo. 5*s.* 6*d.*

OLIVER.—LESSONS IN ELEMENTARY BOTANY. With nearly Two Hundred Illustrations. By DANIEL OLIVER, F.R.S., F.L.S. Third Thousand. 18mo. cloth, 4*s.* 6*d.*

"The manner is most fascinating, and if it does not succeed in making this division of science interesting to every one, we do not think anything can. . . . . Nearly 200 well executed woodcuts are scattered through the text, and a valuable and copious index completes a volume which we cannot praise too highly, and which we trust all our botanical readers, young and old, will possess themselves of."—*Popular Science Review.*

"To this system we now wish to direct the attention of teachers, feeling satisfied that by some such course alone can any substantial knowledge of plants be conveyed with certainty to young men educated as the mass of our medical students have been. We know of no work so well suited to direct the botanical pupil's efforts as that of Professor Oliver's, who, with views so practical and with great knowledge too, can write so accurately and clearly."—*Natural History Review.*

ROSCOE.—LESSONS in ELEMENTARY CHEMISTRY, Inorganic and Organic. By HENRY ROSCOE, F.R.S., Professor of Chemistry in Owen's College, Manchester. With numerous Illustrations and Chromo-Litho. of the Solar Spectra. Ninth Thousand. 18mo. cloth, 4*s.* 6*d.*

It has been the endeavour of the author to arrange the most important facts and principles of Modern Chemistry in a plain but concise and scientific form, suited to the present requirements of elementary instruction. For the purpose of facilitating the attainment of exactitude in the knowledge of the subject, a series of exercises and questions upon the lessons have been added. The metric system of weights and measures, and the centigrade thermometric scale, are used throughout the work.

"A small, compact, carefully elaborated and well arranged manual."—*Spectator.*

MISCELLANEOUS.

ATLAS of EUROPE. GLOBE EDITION. Uniform in size with Macmillan's Globe Series, containing 48 Coloured Maps, on the same scale Plans of London and Paris, and a copious Index, strongly bound in half-morocco, with flexible back. 9*s*.

NOTICE.—This Atlas includes all the Countries of Europe in a Series of Forty-eight Maps, drawn on the same scale, with an Alphabetical Index to the situation of more than 10,000 Places; and the relation of the various Maps and Countries to each other is defined in a general Key-Map.

The identity of scale in all the Maps facilitates the comparison of extent and distance, and conveys a just impression of the magnitude of different Countries. The size suffices to show the Provincial Divisions, the Railways and Main Roads, the Principal Rivers and Mountain Ranges. As a book it can be opened without the inconvenience which attends the use of a folding map.

"In the series of works which Messrs. Macmillan and Co. are publishing under this general title *(Globe Series)* they have combined portableness with scholarly accuracy and typographical beauty, to a degree that is almost unprecedented. Happily they are not alone in employing the highest available scholarship in the preparation of the most elementary educational works; but their exquisite taste and large resources secure an artistic result which puts them almost beyond competition. This little atlas will be an invaluable boon for the school, the desk, or the traveller's portmanteau."—*British Quarterly Review.*

BATES and LOCKYER.—A CLASS BOOK of GEOGRAPHY, adapted to the recent Programme of the Royal Geographical Society. By H. W. BATES and J. N. LOCKYER, F.R.A.S.
[*In the Press.*

CAMEOS from ENGLISH HISTORY. From Rollo to Edward II. By the Author of "The Heir of Redclyffe." Extra fcap. 8vo. 5*s*.

"Contains a large amount of information in a concentrated form, and so skilfully and well is the adventurous, personal, and dramatic element brought out, that any boy of intelligence will find these narratives as fascinating as the most exciting fiction ever penned."—*London Review.*

EARLY EGYPTIAN HISTORY for the Young. With Descriptions of the Tombs and Monuments. New Edition, with Frontispiece. Fcap. 8vo. 5*s*.

"Written with liveliness and perspicuity."—*Guardian.*

"Artistic appreciation of the picturesque, lively humour, unusual aptitude for handling the childish intellect, a pleasant style, and sufficient learning, altogether free from pedantic parade, are among the good qualities of this volume, which we cordially recommend to the parents of inquiring and book-loving boys and girls."—*Athenæum.*

"This is one of the most perfect books for the young that we have ever seen. We know something of Herodotus and Rawlinson, and the subject is certainly not new to us; yet we read on, not because it is our duty, but for very pleasure. The author has hit the best possible way of interesting any one, young or old."—*Literary Churchman.*

HOLE.—A GENEALOGICAL STEMMA of the KINGS of ENGLAND and FRANCE. By the Rev. C. HOLE. In One Sheet. 1*s.*

— A BRIEF BIOGRAPHICAL DICTIONARY. Compiled and Arranged by CHARLES HOLE, M.A., Trinity College, Cambridge. Second Edition, 18mo., neatly and strongly bound in cloth, 4*s.* 6*d.*

The most comprehensive Biographical Dictionary in English,—containing more than 18,000 names of persons of all countries, with dates of birth and death, and what they were distinguished for.

"An invaluable addition to our manuals of reference, and from its moderate price, it cannot fail to become as popular as it is useful."—*Times.*

"Supplies a universal want among students of all kinds. It is a neat, compact, well printed little volume, which may go into the pocket, and should be on every student's table, at hand, for reference."—*Globe.*

HOUSEHOLD (A) BOOK OF ENGLISH POETRY. Selected and arranged, with Notes, by R. C. TRENCH, D.D., Archbishop of Dublin. Extra fcap. 8vo. 5*s.* 6*d.*

"Remarkable for the number of fine poems it contains that are not found in other collections."—*Express.*

"The selection is made with the most refined taste, and with excellent judgment."—*Birmingham Gazette.*

JEPHSON.—SHAKESPEARE'S TEMPEST. With Glossary and Explanatory Notes. By the Rev. J. M. JEPHSON. 18mo. 1*s.* 6*d.*

"His notes display a thorough familiarity with our older English literature, and his preface is so full of intelligent critical remark, that many readers will wish that it were longer."—*Guardian.*

OPPEN.—FRENCH READER. For the use of Colleges and Schools. Containing a Graduated Selection from Modern Authors in Prose and Verse; and copious Notes, chiefly Etymological. By EDWARD A. OPPEN. Fcap. 8vo. cloth, 4*s.* 6*d.*

"Mr. Oppen has produced a French Reader, which is at once moderate yet full, informing yet interesting, which in its selections balances the moderns fairly against the ancients. . . . . The examples are chosen with taste and skill, and are so arranged as to form a most agreeable course of French reading. An etymological and biographical appendix constitutes a very valuable feature of the work."—*Birmingham Daily Post.*

A SHILLING BOOK of GOLDEN DEEDS. A Reading-Book for Schools and General Readers. By the Author of "The Heir of Redclyffe." 18mo. cloth.

"To collect in a small handy volume some of the most conspicuous of these (examples) told in a graphic and spirited style, was a happy idea, and the result is a little book that we are sure will be in almost constant demand in the parochial libraries and schools for which it is avowedly intended."—*Educational Times.*

A SHILLING BOOK of WORDS from the POETS. By C. M. VAUGHAN. 18mo. cloth.

THRING.—Works by **Edward Thring, M.A.,** Head Master of Uppingham :—

— THE ELEMENTS of GRAMMAR taught in ENGLISH. With Questions. Fourth Edition. 18mo. *2s.*

— THE CHILD'S GRAMMAR. Being the substance of "The Elements of Grammar taught in English," adapted for the use of Junior Classes. A New Edition. 18mo. *1s.*

The author's effort in these two books has been to point out the broad, beaten, every-day path, carefully avoiding digressions into the bye-ways and eccentricities of language. This work took its rise from questionings in National Schools, and the whole of the first part is merely the writing out in order the answers to questions which have been used already with success. Its success, not only in National Schools, from practical work in which it took its rise, but also in classical schools, is full of encouragement.

— SCHOOL SONGS. A collection of Songs for Schools. With the Music arranged for Four Voices. Edited by the Rev. E. THRING and H. RICCIUS. Folio. *7s. 6d.*

DIVINITY.

EASTWOOD.—THE BIBLE WORD BOOK. A Glossary of Old English Bible Words. By J. EASTWOOD, M.A., of St. John's College, and W. ALDIS WRIGHT, M.A., Trinity College, Cambridge. 18mo. *5s. 6d.*

HARDWICK.—A HISTORY of the CHRISTIAN CHURCH. MIDDLE AGE. From Gregory the Great to the Excommunication of Luther. By ARCHDEACON HARDWICK. Edited by FRANCIS PROCTER, M.A. With Four Maps constructed for this work by A. KEITH JOHNSTON. Second Edition. Crown 8vo. *10s. 6d.*

— A HISTORY of the CHRISTIAN CHURCH during the REFORMATION. By ARCHDEACON HARDWICK. Revised by FRANCIS PROCTER, M.A. Second Edition. Crown 8vo. *10s. 6d.*

MACLEAR.—Works by the **Rev. G. F. Maclear, B.D.,** Head Master of King's College School, and Preacher at the Temple Church:—

— A CLASS-BOOK of OLD TESTAMENT HISTORY. Fourth Edition, with Four Maps. 18mo. cloth, 4*s*. 6*d*.

"A work which for fulness and accuracy of information may be confidently recommended to teachers as one of the best text-books of Scripture History which can be put into a pupil's hands."—*Educational Times.*

— A CLASS-BOOK of NEW TESTAMENT HISTORY: including the Connection of the Old and New Testament. With Four Maps. Second Edition. 18mo. cloth. 5*s*. 6*d*.

"Mr. Maclear has produced in this handy little volume a singularly clear and orderly arrangement of the Sacred Story.... His work is solidly and completely done."—*Athenæum.*

— A SHILLING BOOK of OLD TESTAMENT HISTORY, for National and Elementary Schools. With Map. 18mo. cloth.

— A SHILLING BOOK of NEW TESTAMENT HISTORY, for National and Elementary Schools. With Map. 18mo. cloth.

— CLASS BOOK of the CATECHISM of the CHURCH of ENGLAND. Second Edition. 18mo. cloth, 2*s*. 6*d*. A Sixpenny Edition in the Press.

PROCTER.—A HISTORY of the BOOK of COMMON PRAYER: with a Rationale of its Offices. By FRANCIS PROCTER, M.A. Seventh Edition, revised and enlarged. Crown 8vo. 10*s*. 6*d*.

In the course of the last twenty years the whole question of Liturgical knowledge has been reopened with great learning and accurate research, and it is mainly with the view of epitomizing their extensive publications, and correcting by their help the errors and misconceptions which had obtained currency, that the present volume has been put together.

— AN ELEMENTARY HISTORY of the BOOK of COMMON PRAYER. By FRANCIS PROCTER, M.A. Second Edition. 18mo. 2*s*. 6*d*.

The author having been frequently urged to give a popular abridgment of his larger work in a form which should be suited for use in schools and for general readers, has attempted in this book to trace the History of the Prayer-Book, and to supply to the English reader the general results which in the larger work are accompanied by elaborate discussions and references to authorities indispensable to the student. It is hoped that this book may form a useful manual to assist people generally to a more intelligent use of the Forms of our Common Prayer.

PSALMS of DAVID Chronologically Arranged. By FOUR FRIENDS. An amended version, with Historical Introduction and Explanatory Notes. Crown 8vo., 10*s*. 6*d*.

"It is a work of choice scholarship and rare delicacy of touch and feeling." —*British Quarterly.*

RAMSAY.—THE CATECHISER'S MANUAL; or, the Church Catechism illustrated and explained, for the use of Clergymen, Schoolmasters, and Teachers. By ARTHUR RAMSAY, M.A. Second Edition. 18mo. 1*s.* 6*d.*

SIMPSON.—AN EPITOME of the HISTORY of the CHRISTIAN CHURCH. By WILLIAM SIMPSON, M.A. Fourth Edition. Fcap. 8vo. 3*s.* 6*d.*

SWAINSON.—A HAND-BOOK to BUTLER'S ANALOGY. By C. A. SWAINSON, D.D., Norrisian Professor of Divinity at Cambridge. Crown 8vo. 1*s.* 6*d.*

WESTCOTT.—A GENERAL SURVEY of the HISTORY of the CANON of the NEW TESTAMENT during the First Four Centuries. By BROOKE FOSS WESTCOTT, B.D., Assistant Master at Harrow. Second Edition, revised. Crown 8vo. 10*s.* 6*d.*

The Author has endeavoured to connect the history of the New Testament Canon with the growth and consolidation of the Church, and to point out the relation existing between the amount of evidence for the authenticity of its component parts and the whole mass of Christian literature. Such a method of inquiry will convey both the truest notion of the connexion of the written Word with the living Body of Christ, and the surest conviction of its divine authority.

— INTRODUCTION to the STUDY of the FOUR GOSPELS. By BROOKE FOSS WESTCOTT, B.D. Third Edition. Crown 8vo. 10*s.* 6*d.*

This book is intended to be an Introduction to the *Study* of the Gospels. In a subject which involves so vast a literature much must have been overlooked; but the author has made it a point at least to study the researches of the great writers, and consciously to neglect none.

— THE BIBLE in the CHURCH. A Popular Account of the Collection and Reception of the Holy Scriptures in the Christian Churches. Second Edition. By BROOKE FOSS WESTCOTT, B.D. 18mo. cloth, 4*s.* 6*d.*

"Mr. Westcott has collected and set out in a popular form the principal facts concerning the history of the Canon of Scripture. The work is executed with Mr. Westcott's characteristic ability."—*Journal of Sacred Literature.*

WILSON.—AN ENGLISH HEBREW and CHALDEE LEXICON and CONCORDANCE to the more Correct Understanding of the English translation of the Old Testament, by reference to the Original Hebrew. By WILLIAM WILSON, D.D., Canon of Winchester, late Fellow of Queen's College, Oxford. Second Edition, carefully Revised. 4to. cloth, 25*s.*

The aim of this work is, that it should be useful to Clergymen and all persons engaged in the study of the Bible, even when they do not possess a knowledge of Hebrew; while able Hebrew scholars have borne testimony to the help that they themselves have found in it.

BOOKS ON EDUCATION.

ARNOLD.—A FRENCH ETON; or, Middle-Class Education and the State. By MATTHEW ARNOLD. Fcap. 8vo. cloth. 2*s*. 6*d*.

"A very interesting dissertation on the system of secondary instruction in France, and on the advisability of copying the system in England."—*Saturday Review*.

— SCHOOLS and UNIVERSITIES on the CONTINENT. 8vo. 10*s*. 6*d*.

BLAKE.—A VISIT to some AMERICAN SCHOOLS and COLLEGES. By SOPHIA JEX BLAKE. Crown 8vo. cloth. 6*s*.

"Miss Blake gives a living picture of the schools and colleges themselves, in which that education is carried on."—*Pall-Mall Gazette*.

"Miss Blake has written an entertaining book upon an important subject; and while we thank her for some valuable information, we venture to thank her also for the very agreeable manner in which she imparts it."—*Athenæum*.

"We have not often met with a more interesting work on education than that before us."—*Educational Times*.

ESSAYS ON A LIBERAL EDUCATION. By CHARLES STUART PARKER, M.A., HENRY SIDGWICK, M.A., LORD HOUGHTON, JOHN SEELEY, M.A., REV. F. W. FARRAR, M.A., F.R.S., &c., E. E. BOWEN, M.A., F.R.A.S., J. W. HALES, M.A., J. M. WILSON, M.A., F.G.S., F.R.A.S., W. JOHNSON, M.A. Edited by the Rev. F. W. FARRAR, M.A., F.R.S., late Fellow of Trinity College, Cambridge; Fellow of King's College, London; Assistant Master at Harrow; Author of "Chapters on Language," &c., &c. Second Edition. 8vo. cloth, 10*s*. 6*d*.

FARRAR.—ON SOME DEFECTS IN PUBLIC SCHOOL EDUCATION. A Lecture delivered at the Royal Institution. With Notes and Appendices. Crown 8vo. 1*s*.

THRING.—EDUCATION AND SCHOOL. By the Rev. EDWARD THRING, M.A., Head Master of Uppingham. Second Edition. Crown 8vo. cloth. 6*s*.

YOUMANS.—MODERN CULTURE: its True Aims and Requirements. A Series of Addresses and Arguments on the Claims of Scientific Education. Edited by EDWARD L. YOUMANS, M.D. Crown 8vo. 8*s*. 6*d*.

CAMBRIDGE:—PRINTED BY JONATHAN PALMER.